Section 5.5 Complex Eigenvalues

Recall – Complex numbers:

a. A complex number is a number written in the form

$$z = a + bi$$

where a and b are real numbers and i is a formal symbol satisfying the relation $i^2 = -1$. a is called the real part of z, denoted by Re z, and b is the imaginary part of z denoted by Im z.

$\mathbb{C}$

b. The complex number system, denoted by $\mathbb{C}$ is the set of all complex numbers, together with the following operations of addition and multiplication

$(a+bi)(c+di)$
$= ac + adi + bci + bdi^2$
$i^2=-1$ $= ac + (ad+bc)i + -1 \cdot bd$
$= (ac-bd) + (ad+bc)i$

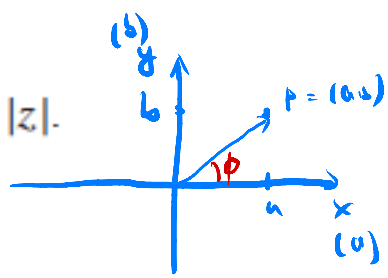
$$(a+bi)+(c+di) = (a+c)+(b+d)i$$
$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$

c. The conjugate of $z = a+ib$ is $\bar{z} = a - bi$.

d. The absolute value or modulus of z is $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$.

e. Properties of conjugates and absolute value:

1. $\overline{z} = z$ if and only if z is a real number.
2. $\overline{w+z} = \overline{w} + \overline{z}$.
3. $\overline{wz} = \overline{w}\,\overline{z}$; in particular, $\overline{rz} = r\overline{z}$ if r is a real number.
4. $z\overline{z} = |z|^2 \geq 0$.
5. $|wz| = |w||z|$.
6. $|w+z| \leq |w| + |z|$.



f. Geometric interpretation.

g. Polar coordinates for a nonzero complex number $z = a+bi$: let ϕ $(-\pi < \phi \leq \pi)$ be the angle between the positive real axis and the point (a,b), we call it the argument of z. Then $z = |z|(\cos\phi + i\sin\phi)$

$\| \sqrt{a^2+b^2}$

Real and Imaginary Parts of Vectors:

Let $\mathbf{x} \in \mathbb{C}^n$, denote the real part of $\mathbf{x}$ by Re$\mathbf{x}$ and the imaginary part of $\mathbf{x}$ by Im$\mathbf{x}$.

Example 1: Let $\mathbf{x} = \begin{bmatrix} 1+i \\ 2 \\ 3-2i \end{bmatrix}$, find Re$\mathbf{x}$ and Im$\mathbf{x}$.

$$\vec{x} = \begin{pmatrix} 1+i \\ 2+0i \\ 3-2i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} i \\ 0 \cdot i \\ -2i \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

$\text{Real}(\vec{x}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$\text{Img}(\vec{x}) = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$$\bar{\vec{x}} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} i = \begin{pmatrix} 1-i \\ 2 \\ 3+2i \end{pmatrix}$$ conjugate of $\vec{x}$.

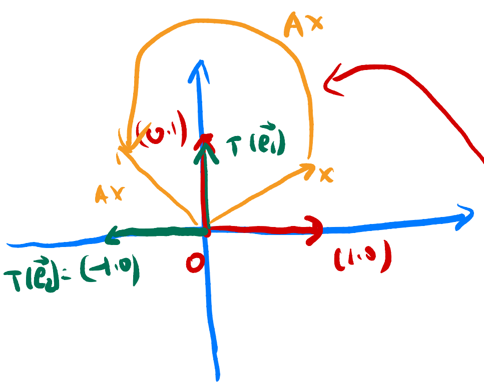
Example 2: If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, find the complex eigenvalues of A and corresponding eigenvectors.

Remark: The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ on $\mathbb{R}^2$ rotates the plane counterclockwise through a quarter-turn. Obviously, no nonzero vector is mapped into a multiple of itself, so A has no eigenvectors in $\mathbb{R}^2$ and hence no real eigenvalues.

For any linear transformation T. there exists A s.t. $T(x) = A \cdot x$

$A = [\,T(\vec{e}_1), T(\vec{e}_2)\,]$, $\quad \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $T(\vec{e}_1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $T(\vec{e}_2) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$



If real eigenvalue exists

$$\underbrace{Ax}_{\text{rotation of } x} = \underbrace{\lambda x}_{\text{scaling of } x \text{ by } \lambda}$$

rotation of x $\neq$ scaling of x by λ

$\Rightarrow$ the real eigenvalue does not exist.

Solve $\det(A - \lambda I) = 0$

$$\det\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

$$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$\Rightarrow \quad \lambda^2 = -1$$

Replace -1 with i^2 $(i^2 = -1)$

$$\lambda^2 = i^2$$

$$\Rightarrow \quad \lambda = \pm i$$

$\lambda = +i$ Find $x^{\neq 0}$ such that

$$(A - \lambda I) \cdot x = 0$$

$$\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\begin{cases} -ia - b = 0 & (1) \\ a - ib = 0 & (2) \end{cases}$$

Since λ is an eigvalue of A,

$\Rightarrow$ you can always find a non-zero x s.t. $(A-\lambda I)x = 0$

pick up ① $-ia - b = 0$

$$b = -ia.$$

set $a = 1$, $\Rightarrow b = -i$

$\Rightarrow$ basis of eig-space $\vec{v}_1 = \underbrace{\begin{pmatrix} 1 \\ -i \end{pmatrix}}_{\text{eigen vector of } \lambda = i}$

$\lambda = -i.$ $\quad (A - \lambda I)v = 0$

$$\begin{pmatrix} +i & -1 \\ 1 & +i \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\begin{cases} ai - b = 0 \\ a + ib = 0 \end{cases}$$

$ai = b$, set $a = 1$, $\Rightarrow b = i$

eigen-vector $\vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

Eigenvalues and Eigenvectors of a Real Matrix That Acts on $\mathbb{C}^n$:

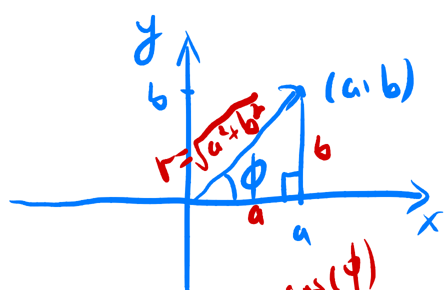
Let A be an $n \times n$ matrix with **real entries**. If λ is an eigenvalue of A and $\mathbf{x}$ is a corresponding eigenvector in $\mathbb{C}^n$, then

suppose λ is a complex eigenvalue of A with eigen vector x.

then $\bar{\lambda}$ is still an eigenvalue & its eigen vector is $\bar{x}$.

Remark: When A is real, its complex eigenvalues occur in conjugate pairs.

Geometric explanation: If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$ and both nonzero. Then the eigenvalues of C are $\lambda = a \pm bi$.



If $r = |\lambda| = \sqrt{a^2+b^2}$, then

$$C = r\begin{bmatrix} \overset{\cos(\phi)}{a/r} & -b/r \\ b/r & a/r \end{bmatrix} = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \cdot \underbrace{\begin{pmatrix} \cos(\phi), & -\sin\phi \\ \sin(\phi), & \cos(\phi) \end{pmatrix}}$$

→ rotate the vector by an angle ϕ

→ scaling by $r = \sqrt{a^2+b^2}$

$$\det(C - \lambda I) = 0$$

$$\det\begin{pmatrix} a-\lambda, & -b \\ b, & a-\lambda \end{pmatrix} = 0$$

$$(a-\lambda)^2 = -b^2$$

replace -1 by i^2 ($i^2 = -1$)

$$(a-\lambda)^2 = \underbrace{i^2}_{-1} b^2$$

$$a-\lambda = \pm ib$$

$$\lambda = a \pm bi$$

$\lambda = a + bi$

$$(C - \lambda I) \cdot v = 0$$

$$\begin{pmatrix} -bi & -b \\ b & -bi \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$-bix - by = 0$$

$$bx - biy = 0$$

$$-bix = by \Rightarrow -ix = y$$

set $x = 1 \Rightarrow y = -i$ $\quad \vec{v_1} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$\lambda = a - bi \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{real}(\vec{v}_1)} + i \underbrace{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}_{\text{img}(\vec{v}_1)}$$

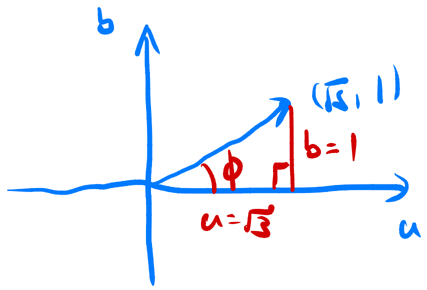
$$\vec{v} = \text{real}(\vec{v}_1) - \text{img}(\vec{v}_1) \cdot i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix}. \quad //$$

$a = \sqrt{3}$
$b = +1$

Example 4: List the eigenvalues of $A = \begin{bmatrix} \sqrt{3} & \cancel{3\sqrt{3}}\ -1 \\ +1\ \cancel{-3\sqrt{3}} & \sqrt{3} \end{bmatrix}$. Note the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is the composition of a rotation and a scaling. Give the angle ϕ of the rotation, where $-\pi < \phi \leq \pi$, and give the scale factor r.

eigenvalue: $a \pm bi = \sqrt{3} \pm i$



$\tan(\phi) = \frac{b}{a} = \frac{1}{\sqrt{3}}$

$\phi = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$r = \sqrt{a^2+b^2} = 2.$

Theorem: Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and an associated eigenvector $\mathbf{v}$ in $\mathbb{C}^2$. Then

$$A = PCP^{-1}, \text{ where } P = [Re\mathbf{v} \quad Im\mathbf{v}] \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

A is similar to C.