Section 5.7 Applications to Differential Equations

Consider a system of ordinary **differential equations**: (ODEs)

$$
\begin{aligned}
x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n
\end{aligned}
\tag{1}
$$

① $x_1 \dots x_n$, they are all function of t.

② $a_{11} \dots a_{nn}$ are all const.

③ $x_i' = \frac{d x_i(t)}{dt} = \dot{x}_i$

We can write the system as a matrix differential equation

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

where

$$x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

A solution of equation (1) is a vector-valued function that satisfies (1) for all t in some interval of real numbers, such as $t \geq 0$.

where c & d are arbitrary real numbers.

Superposition: : If $\mathbf{u}$ and $\mathbf{v}$ are solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$, then $c\mathbf{u} + d\mathbf{v}$ is also a solution.

thm: $x' = Ax$, $A \in \mathbb{R}^{n \times n}$, the system has n- linearly indep solutions $\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n$

def: $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is called fundamental set of solutions.

Fundamental set of solutions to (1):

If A is $n \times n$, then there are n linearly independent functions in a fundamental set, and each solution of (1) is a unique linear combination of these n functions.

any solution u of $x' = Ax$, $u = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_n\vec{v}_n$, $c_1 \dots c_n \in \mathbb{R}$

That is, a fundamental set of solutions is a basis for the set of all solutions of (1), and the solution set is an n-dimensional vector space of functions.

def: general solution can be written as $y = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n$, $c_1 \dots c_n \in \mathbb{R}$.

Initial value problem: . If a vector $\mathbf{x}_0$ is specified, then the initial value problem is to construct the (unique) function $\mathbf{x}$ such that

IVP

$$
\begin{aligned}
\mathbf{x}'(t) &= A\mathbf{x}(t) \\
\mathbf{x}(0) &= \mathbf{x}_0 \leftarrow \text{initial condition (IC)}
\end{aligned}
$$

$\hookrightarrow t = 0$

Example 1: Consider $\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Here the matrix A is diagonal, we call the system *decoupled*. Find solutions of this system.

$$\begin{cases} x_1'(t) = 3x_1(t) & \text{①} \\ x_2'(t) = -5x_2(t) & \text{②} \end{cases}$$

x_1 & x_2 do not depend on each.

x_1: guess $x_1 = e^{rt}$, r is unknown.

Target: find r.

Remark:

substitute e^{rt} into ① & solve for r.

$$(e^{rt})' = 3e^{rt}$$

$$r\cancel{e^{rt}} = 3\cancel{e^{rt}}$$

$$r = 3$$

$\Rightarrow$ e^{3t} is a solution of $x_1(t)$.

Example 2: The circuit in Figure can be described by the differential equation

initial condition $x_1(0) = 5$

$x_2(0) = 4$

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -(1/R_1 + 1/R_2)/C_1 & 1/(R_2C_1) \\ 1/(R_2C_2) & -1/(R_2C_2) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

where $x_1(t)$ and $x_2(t)$ are the voltages across the two capacitors at time t. Suppose resistor R_1 is 1 ohm, R_2 is 2 ohms, capacitor C_1 is 1 farad, and C_2 is .5 farad, and suppose there is an $\boxed{\text{initial}}$ charge of 5 volts on capacitor C_1 and 4 volts on capacitor C_2. Find formulas for $x_1(t)$ and $x_2(t)$ that describe how the voltages change over time.

cont'd

$x_2 = e^{rt}$, r is unknown.

substitute x_2 into ②

$$(e^{rt})' = -5e^{rt}$$

$$r\cancel{e^{rt}} = -5\cancel{e^{rt}}$$

$$r = -5.$$

$\Rightarrow$ e^{-5t} is a solution for $x_2(t)$.

In general, A is not decoupled.

(3) $x' = Ax$,

construct the solution as $\vec{x} = \underbrace{\vec{v}}_{\text{const.}} \underbrace{\boxed{e^{rt}}}_{\text{scalar depends on } t}$

$\vec{v}$ & r are unknown.

substitute $\vec{x} = \vec{v} e^{rt}$ into (3)

$$r \cancel{e^{rt}} \vec{v} = \cancel{e^{rt}} A \vec{v}$$

$$A \vec{v} = r \vec{v}$$

$\Rightarrow$ $(r, \vec{v})$ is an eig-pair of A.

In this course, we focus on A which has n distinct eig-values.

thm: each eig-pair $(\lambda_i, \vec{v}_i)$ will give us one linearly indep. solution.

$$\vec{v}_i e^{\lambda_i t}$$

Eg 2.

$$A = \begin{pmatrix} -1.5 & 0.5 \\ 1 & -1 \end{pmatrix}. \quad x(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}.$$

$$\begin{cases} \lambda_1 = -0.5 \\ \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{cases} \qquad \begin{cases} \lambda_2 = -2 \\ \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{cases}$$

general solution,

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t},$$

c_1 & c_2 are constants.

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-0.5t} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \qquad (4)$$

we know $x(0) = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$. substitute $t = 0$ into (4)

$$\underbrace{\begin{pmatrix} 5 \\ 4 \end{pmatrix}}_{\text{given}} = \underbrace{c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-0.5 \cdot 0} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{\text{substitute } t=0 \text{ into the general sol.}} e^{-2 \cdot 0}$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

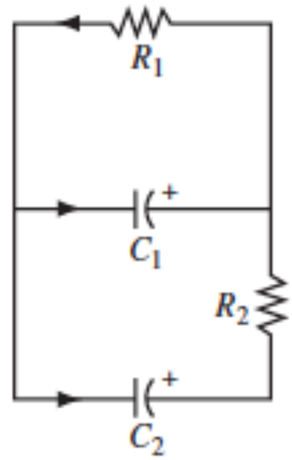
$c_1 = 3$ & $c_2 = -2$.

Solution:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \vec{x}(t) = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-0.5t} + (-2)\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t}$$

OR.

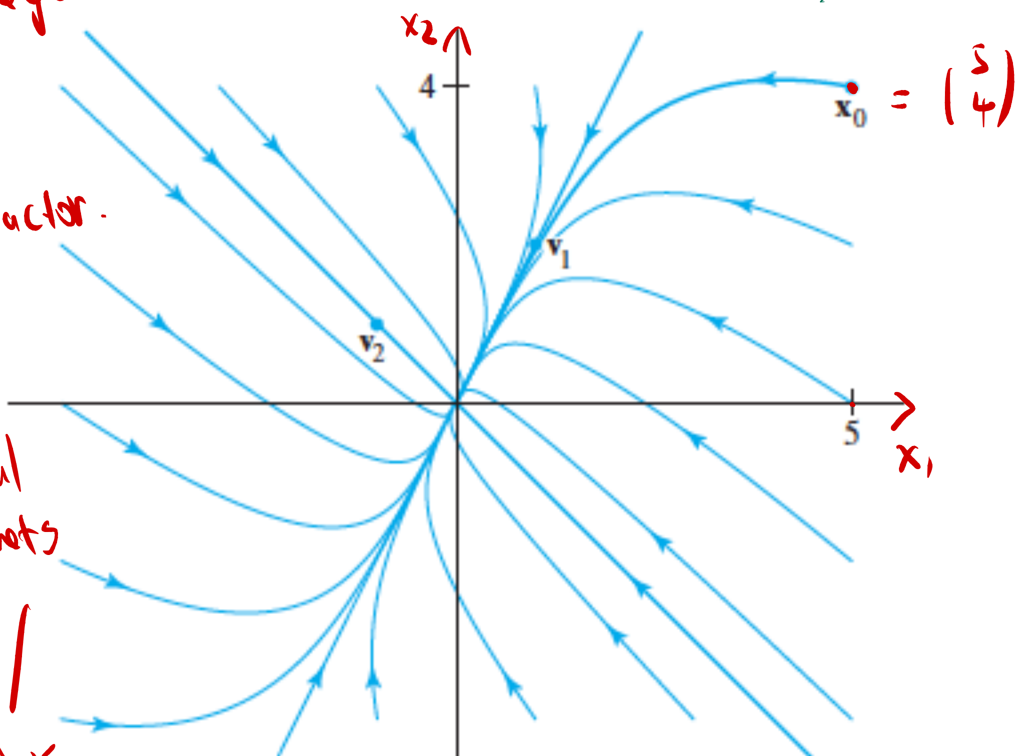
$$\begin{cases} x_1(t) = 3e^{-0.5t} + 2e^{-2t} \\ x_2(t) = 6e^{-0.5t} - 2e^{-2t} \end{cases}$$



Remark: In Figure , the origin is called an **attractor** or **sink** of the dynamical system because all trajectories are drawn into the origin.
If the eigenvalues in Example 2 were positive instead of negative, the corresponding trajectories would be similar in shape, but the trajectories would be traversed away from the origin. In such a case, the origin is called a **repeller**, or **source**, of the dynamical system.

phase diagram.

① distinct negative real e.values,
→ sink/attractor.



② distinct positive real eigen values
→ source/repellor.

Example 3: Suppose a particle is moving in a planar force field and its position vector $\mathbf{x}$ satisfies $\mathbf{x}'(t) = A\mathbf{x}(t)$ and $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $A = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 2.9 \\ 2.6 \end{bmatrix}$ Solve this initial value problem for $t \geq 0$, and sketch the trajectory of the particle.

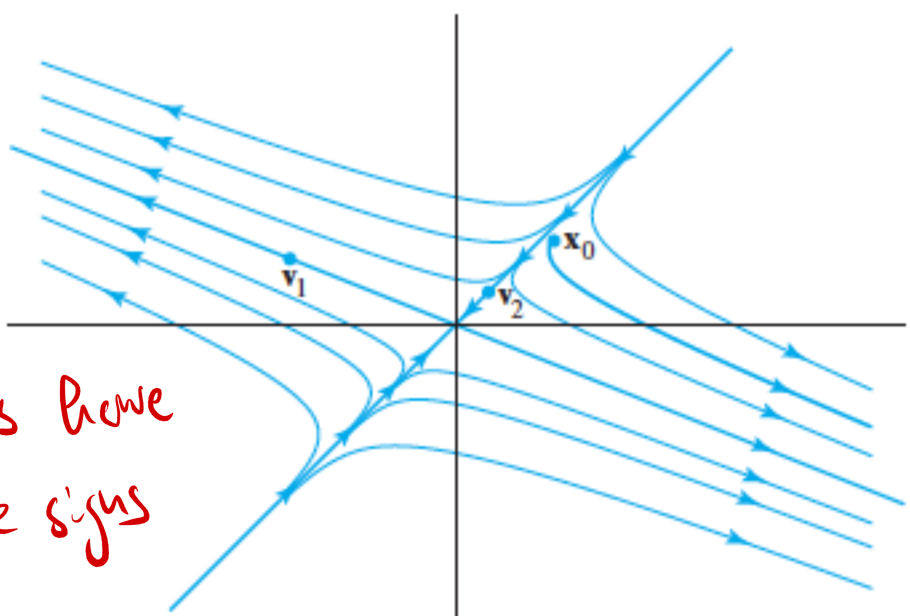
$$\begin{cases} \lambda_1 = 6, & \vec{v}_1 = \begin{pmatrix} -5 \\ 2 \end{pmatrix} \\ \lambda_2 = -1 & \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{cases}$$

general solution: $y = c_1 \cdot \underline{\vec{v}_1 e^{\lambda_1 t}} + c_2 \cdot \underline{\vec{v}_2 \cdot e^{\lambda_2 t}}$, c_1 & c_2 are constants.

Remark: In Figure , the origin is called an **saddle point** of the dynamical system because some trajectories approach the origin at first and then change direction and move away from the origin. A saddle point arises whenever the matrix A has both positive and negative eigenvalues.

③ If the real eigenvalues have opposite signs

$\Rightarrow$ saddle point.



$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \vec{y} = c_1 \begin{pmatrix} -5 \\ 2 \end{pmatrix} e^{6t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

$$\begin{cases} x_1(t) = -5 c_1 e^{6t} + c_2 e^{-t} \\ x_2(t) = 2 c_1 e^{6t} + c_2 e^{-t} \end{cases} \quad (5)$$

I.V.P. (solve for c_1 & c_2)

substitute $t=0$ into (5) (general solution).

$$x_1(0) = 2.4 = -5c_1 + c_2$$

$$x_2(0) = 2.6 = 2c_1 + c_2$$

$$\Rightarrow \begin{cases} c_1 = -3/70 \\ c_2 = 188/70 \end{cases}$$

$$\begin{cases} x_1(t) = +\frac{3}{70} \cdot 5 e^{6t} + \frac{188}{70} e^{-t} \\ x_2(t) = -\frac{3}{70} \cdot 2 e^{6t} + \frac{188}{70} e^{-t} \end{cases}$$

Decoupling a dynamical system : For any dynamical system $\mathbf{x}'(t) = A\mathbf{x}(t)$, if A is diagonalizable, i.e. $A = PDP^{-1}$.

Example 4: Make a change of variable that decouples the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$. Write the equation $\mathbf{x}(t) = P\mathbf{y}(t)$ that leads to the uncoupled system $\mathbf{y}'(t) = D\mathbf{y}(t)$, specifying P and D. $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$

Complex eigenvalues : Let λ and $\overline{\lambda}$ be a pair of complex eigenvalues of A, associated with complex eigenvectors $\mathbf{v}$ and $\overline{\mathbf{v}}$. So two solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are

$$\vec{v}e^{\lambda t} \quad \& \quad \vec{\overline{v}}\, e^{\overline{\lambda} t}$$

eigenvalue $\lambda = a + bi$, v is the corresponding complex e-vector.

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \vec{y}_1 = [\mathrm{Re}(v)\cos(bt) - \mathrm{Im}(v)\sin(bt)]\, e^{at}$$

$$\vec{y}_2 = [\mathrm{Re}(v)\sin(bt) + \mathrm{Im}(v)\cos(bt)]\, e^{at}$$

$\vec{y}_1$ & $\vec{y}_2$ are 2 linearly indep solutions.

Example 5: Construct the general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ involving complex eigenfunctions and then obtain the general real solution. Describe the shapes of typical trajectories. $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$

$$\det \begin{pmatrix} -3-\lambda & 2 \\ -1 & -1-\lambda \end{pmatrix} = 0$$

$$(\lambda+3)(\lambda+1) + 2 = 0$$

$$\lambda^2 + \lambda + 3\lambda + 3 + 2 = 0$$

$$\underbrace{\lambda^2 + 4\lambda + 4}\, + 1 = 0$$

Complex eigenvalues : Let λ and $\overline{\lambda}$ be a pair of complex eigenvalues of A, associated with complex eigenvectors $\mathbf{v}$ and $\overline{\mathbf{v}}$. So two solutions of $\mathbf{x}'(t) = A\mathbf{x}(t)$ are

$$(\lambda+2)^2 = -1$$

$$\lambda = a + bi$$

$$\overline{\lambda} = a - bi$$

$$i^2 = -1$$

$$(\lambda+2)^2 = i^2$$

$$\lambda + 2 = \pm i$$

$$\lambda = -2 \pm i$$

Example 5: Construct the general solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ involving complex eigenfunctions and then obtain the general real solution. Describe the shapes of typical trajectories. $A = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

real(λ) img(λ)

$\lambda = -2 + i$, $a = -2$, $b = 1$

$$(A - \lambda I) \cdot w = 0$$

$$\begin{pmatrix} -1-i & 2 \\ -1 & 1-i \end{pmatrix} \cdot \begin{pmatrix} p \\ q \end{pmatrix} = 0$$

From the 2^{nd} row

$$\Rightarrow \quad -p + (1-i)q = 0$$

$$p = (1-i)q$$

$$i\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\Vert$$

set $q = 1$, $p = 1-i$

eigen vector $\vec{v} = \begin{pmatrix}1-i\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}-i\\0\end{pmatrix}$

$$\mathrm{re}(\vec{v}) = \begin{pmatrix}1\\1\end{pmatrix} \qquad \mathrm{img}(\vec{v}) = \begin{pmatrix}-1\\0\end{pmatrix}$$

$\bar{v} = \begin{pmatrix}1+i\\1\end{pmatrix}$ is the e-vector of $\bar{\lambda}$.

general sol:

$\vec{y}(t) = c_1\vec{v}e^{\lambda t} + c_2\vec{\bar{v}}e^{\bar{\lambda} t}$, c_1 & c_2 are const.

real solution: $\lambda = -2+i$, v

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \vec{y}_1 = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin(t) \right] e^{-2t}$$

$$\vec{y}_2 = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos(t) \right] e^{-2t}$$

$\vec{y}_1$ & $\vec{y}_2$ are 2 linearly indep. solutions.

$\star$. you can also construct the solution

using $\bar{v}$ & $\bar{\lambda}$

$$A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix} \quad \text{solve} \quad X' = AX.$$

$$\begin{cases} \lambda = -2+i \\ v = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} \end{cases} \qquad \begin{cases} \bar{\lambda} = -2-i \\ \bar{v} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \end{cases}$$

complex general solution

$$\vec{y} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = c_1 v e^{-2+i} + c_2 \bar{v} e^{-2-i}$$

pick up $\begin{cases} \lambda = -2+i \\ v = \begin{pmatrix} 1-i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} -i \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix} \end{cases}$

$$a = -2 \quad (\operatorname{Re}(\lambda)) \qquad \operatorname{Re}(v) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$b = 1 \quad (\operatorname{img}(\lambda)) \qquad \operatorname{img}(v) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \vec{y}_1(t) = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(t) - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \sin(t) \right] e^{-2t}$$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \vec{y}_2(t) = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(t) + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \cos(t) \right] e^{-2t}$$

general solution

$\vec{y}(t) = c_1 \vec{y}_1(t) + c_2 \vec{y}_2(t)$, c_1 & c_2 are constants.

$$\vec{y}(t) = c_1 e^{-2t} \cdot \begin{pmatrix} \cos(t) + \sin(t) \\ \cos(t) \end{pmatrix}$$

$$+ c_2 e^{-2t} \cdot \begin{pmatrix} \sin(t) - \cos(t) \\ \sin(t) \end{pmatrix}. \quad //$$

1. complex eigenvalues → spiral
2. distinct neg real e.vals sink/attractor
3. distinct positive real e.vals , source/repeller.
4. real e.vals with opposite signs saddle point.