Section 6.1 Inner Product, Length, Orthogonality

Definition: Let **u** and **v** be vectors in $\mathbb{R}^n$,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the inner product, also referred to as a dot product, of **u** and **v** is

$$u \cdot v = \underbrace{u^t v}_{\text{std matrix multiplication}} = [u_1\ u_2\ \cdots\ u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n \in \mathbb{R}$$

Theorem: Let **u**, **v** and **w** be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ $(AB \neq BA)$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$

$u \cdot u = u^t u$
$= u_1^2 + u_2^2 + \cdots + u_n^2$

Definition: The length (or norm) of a **v** is the nonnegative scaler $||\mathbf{v}||$ defined by

$$||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}, \text{ and } ||\mathbf{v}||^2 = \mathbf{v} \cdot \mathbf{v}$$

For any scalar c,

$$||c\mathbf{v}|| \overset{*}{=} |c|\,||\mathbf{v}||$$

Example 1: Find a unit vector in the direction of $\vec{u} = \begin{bmatrix} -2 \\ 4 \\ -3 \end{bmatrix}$

$\vec{u} \in \mathbb{R}^n$. a vector of length (norm) = 1.

unit vector in the direction of $\vec{u}$

$\frac{\vec{u}}{||\vec{u}||}$

$\left|\left| \frac{\vec{u}}{||\vec{u}||} \right|\right| = \left| \frac{1}{||\vec{u}||} \right| \cdot ||\vec{u}|| = 1$

$||u|| = \sqrt{(-2)^2 + 4^2 + (-3)^2} = \sqrt{29}$

$\frac{u}{||u||} = \frac{1}{\sqrt{29}} \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$

Definition: For **u** and **v** in $\mathbb{R}^n$, the distance between **u** and **v**, written as dist(**u**, **v**), is the length of the vector $\mathbf{u} - \mathbf{v}$,

$$\text{dist}(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$$

$dist(u,v) = ||u - v||$
$dist(v,u) = ||v - u||$ $\Rightarrow$?

$||u - v|| = ||-(v - u)||$
$= |-1| \cdot ||v - u||$

Example 2: Find the distance between $\mathbf{x} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$dist(x, y) = \|x - y\|$

$= \left\| \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\| = \sqrt{3^2 + 7^2} = \sqrt{58}$

$u^t v$

$\|$

Orthogonal vectors: Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal to each other if $\mathbf{u} \cdot \mathbf{v} = 0$. Zero vector is orthogonal to every vector in $\mathbb{R}^n$.

The PythagoreanTheorem: Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal to each other if and only if $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$.

(1) ① def.

$x \cdot y = x^t y = 12 \cdot 2 + 3 \cdot (-3) + (-5) \cdot 3$

$= 0 \Rightarrow x$ & y are orthogonal to each other.

Example 3: Determine which pairs of vectors are orthogonal

$(1)\mathbf{x} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$

② $\overset{\text{thm}}{x + y} = \begin{pmatrix} 14 \\ 0 \\ -2 \end{pmatrix}$

check if $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

$\|x+y\|^2 = 14^2 + 0^2 + (-2)^2$

$\|x\|^2 = 12^2 + 3^2 + (-5)^2$

$\|y\|^2 = 2^2 + (-3)^2 + 3^2$

✓

$(2)\mathbf{x} = \begin{bmatrix} -3 \\ 7 \\ 4 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ -8 \\ 15 \\ -7 \end{bmatrix}$

$w^{\perp} = \{v \in \mathbb{R}^n, v \cdot y = 0, y \in W\}$

Definition: If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be orthogonal to W. The set of all vectors that are orthogonal to W is called the orthogonal component of W and is denoted by $W^{\perp}$. (perpendicular)

(complement)

1. A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to every vector in a set that spans W.
2. $W^{\perp}$ is a subspace of $\mathbb{R}^n$.

Theorem: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A, and the orthogonal complement of the column space of A is the null space of A^T:

$$(\mathrm{Row}A)^{\perp} = \mathrm{Nul}A \quad \text{and} \quad (\mathrm{Col}A)^{\perp} = \mathrm{Nul}A^T$$

Row space of $A \in \mathbb{R}^{m \cdot n}$

$\mathrm{Row}(A)$ = span of all rows of A.

① If $x \in \mathrm{null}(A)$, $Ax = 0$

$\Rightarrow$ $x \cdot v = 0$, for any $v \in \mathrm{Row}(A)$

② $x \cdot v = 0$ for any $v \in \mathrm{Row}(A)$

$\Rightarrow$ $Ax = 0$, or, x is in $\mathrm{null}(A)$.