Section 6.2 Orthogonal Sets

Definition: A set of vectors $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an orthogonal set if each pair of distinct vectors from the set is orthogonal, that is, if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.
Example 1: Determine whether the following set of vectors are orthogonal

check if $u_1 u_2$, $u_2 u_3$, $u_1 u_3$ are all $= 0$

$$\begin{array}{ccc} u_1 & u_2 & u_3 \\ \begin{bmatrix} -1 \\ 4 \\ -3 \end{bmatrix}, & \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}, & \begin{bmatrix} 3 \\ -4 \\ -7 \end{bmatrix} \end{array}$$

Remark: orthogonality $\Rightarrow$ linear indep.
linear indep $\not\Rightarrow$ orthogonal

$u_1 \cdot u_2 = u_1^t u_2 = u_2^t u_1 = (-1)\cdot 5 + 4\cdot 2 + (-3)\cdot 1 = 0$

$u_2 \cdot u_3 = 5\cdot 3 + 2\cdot -4 + 1\cdot -7 = 0$

$u_1 \cdot u_3 = (-1)\cdot 3 + 4\cdot(-4) + (-3)(-7) \neq 0$ $\Rightarrow$ not orthogal

Theorem: If $S = \{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.
Definition: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

↳ if a basis is also orthogal

why orthogonal basis is very useful ?

Theorem: Let $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \underbrace{[u_1 \cdots u_p]}_{A} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_p \end{pmatrix}$$

are given by

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \quad (j = 1, \cdots, p)$$

An Orthogonal Projection: Given a $\boxed{\text{nonzero}}$ vector $\mathbf{u}$ in $\mathbb{R}^n$. Decompose a vector $\mathbf{y}$ in $\mathbb{R}^n$ into the sum of two vectors

any y.

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{1}$$

where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α and $\mathbf{z}$ is some vector orthogonal to $\mathbf{u}$.

Target: need to find α such that the decomposition satisfies, $\hat{y} = \alpha u$, $z \cdot u = 0$.

$z = y - \hat{y} = y - \alpha u \Rightarrow \alpha u \cdot u = y \cdot u$ → If $\alpha = \frac{y \cdot u}{u \cdot u}$, & then $\hat{y} = \frac{y \cdot u}{u \cdot u} u$, $z = y - \hat{y}$

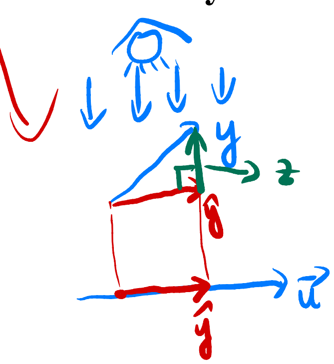
$(y - \alpha u) \cdot u = 0 \Rightarrow \alpha = \frac{y \cdot u}{u \cdot u}$

Thus (1) is satisfied with $\mathbf{z}$ orthogonal to $\mathbf{u}$ if and only if The vector $\hat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto $\mathbf{u}$, and the vector $\mathbf{z}$ is the component of $\mathbf{y}$ orthogonal to $\mathbf{u}$.
The projection is determined by the subspace L spanned by $\mathbf{u}$ (the line through $\mathbf{u}$ and $\mathbf{0}$). Sometimes $\hat{\mathbf{y}}$ is denoted by $\text{proj}_L\mathbf{y}$ and is called the orthogonal projection of $\mathbf{y}$ onto L. That is,

L is the subspace spanned by vector $\vec{u}$.

$$\hat{\mathbf{y}} = \text{proj}_L\mathbf{y} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \tag{2}$$

Example 2: Compute the orthogonal projection of $\underset{y}{\begin{bmatrix} 7 \\ 6 \end{bmatrix}}$ onto the line through $\begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and the origin. → $\vec{u}$
Then write $\mathbf{y}$ as the sum of a vector in Span $\{\mathbf{u}\}$ and a vector orthogonal to $\{\mathbf{u}\}$.



compute $\hat{y}$ given $\vec{u} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$$\text{Proj}_L y = \frac{y \cdot u}{u \cdot u} u = \frac{7 \cdot 4 + 6 \cdot 2}{4 \cdot 4 + 2 \cdot 2} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \frac{40}{20} \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$$

Decomposition: $y = \hat{y} + z$, $\hat{y} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$ satisfy $\hat{y} = \alpha \cdot u$

$z = y - \hat{y} = \begin{pmatrix} 7 \\ 6 \end{pmatrix} - \begin{pmatrix} 8 \\ 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (z will satisfy $z \cdot u = 0$)

Orthonormal Sets: A set $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an orthonormal set if it is an orthogonal set of unit vectors. If W is the subspace spanned by such a set, then $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ in

$\mathbb{R}^n$ is an orthonormal basis for W, since the set is automatically linearly independent.

Example 3: Determine whether the following set of vectors are orthonormal. If a set is only orthogonal, normalize the vectors to produce an orthonormal set.

$$\begin{array}{cc} u_1 & u_2 \\ \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}, & \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix} \end{array}$$

orthogonal: $u_1 \cdot u_2 = 0 \Rightarrow$ orthogonal

$\|u_1\| = \sqrt{(1/3)^2+(1/3)^2+(1/3)^2} = \frac{\sqrt{3}}{3} \neq 1 \Rightarrow$ this NOT orthonormal.

find the unit vector in the direction of u_1, $\rightarrow \hat{u}_1 = \frac{u_1}{\|u_1\|} = \frac{1}{\sqrt{3}/3} \cdot \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}$

— — — — — — — — u_2, $\rightarrow \hat{u}_2 = \frac{u_2}{\|u_2\|}$

$\hat{u}_1 \cdot \hat{u}_2$

$= \frac{u_1}{\|u_1\|} \cdot \frac{u_2}{\|u_2\|}$

$= \frac{1}{\|u_1\|} \cdot \frac{1}{\|u_2\|} \cdot \underbrace{u_1 \cdot u_2}_{=0} = 0$

orthonormal set $\left\{ \frac{u_1}{\|u_1\|}, \frac{u_2}{\|u_2\|} \right\}$

Theorem: An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

$U^{-1} \cdot U = I \Rightarrow U^{-1} = U^T$

Review. ① u & v are orthogonal, $u \cdot v = 0$

② orthogonality $\Rightarrow$ linearly indep.

③ orthonormal = $u \cdot v = 0$ ⊕ $\|u\| = \|v\| = 1$.

Theorem: Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$. Then

linear map $x \rightarrow Ux$ preserves the length of the vector.

a . $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b . $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c . $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = \mathbf{0}$.

the linear map also preserves the orthogonality.