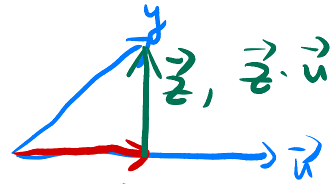
$y = \hat{y} + z$



$\vec{z}$, $\vec{z}\cdot\vec{u} = 0$

Section 6.3 Orthogonal Projections

$\hat{y} = \frac{y\cdot u}{u\cdot u}\vec{u}$ → orthogonal projection of y onto $\vec{u}$

Theorem:

> **The Orthogonal Decomposition Theorem**
>
> Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be written uniquely in the form
>
> $z\cdot u = 0$ for all $u \in W$.
>
> $$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} \tag{1}$$
>
> where $\hat{\mathbf{y}}$ is in W and $\mathbf{z}$ is in $W^{\perp}$. In fact, if $\{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is any orthogonal basis of W, then
>
> $$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \cdots + \frac{\mathbf{y}\cdot\mathbf{u}_p}{\mathbf{u}_p\cdot\mathbf{u}_p}\mathbf{u}_p \tag{2}$$
>
> ↳ orthogonal project of y onto W $\quad \mathrm{Proj}_W\, y = \hat{y}$
>
> and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}}$.

Example 1: Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Remark: If $y \in W$, $\hat{y} = y$, $z = 0$

Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = Span\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.

$$\hat{y} = \frac{y\cdot u_1}{u_1\cdot u_1}u_1 + \frac{y\cdot u_2}{u_2\cdot u_2}u_2$$

$$= \frac{2+10-3}{4+25+1}\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \frac{-2+2+3}{4+1+1}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

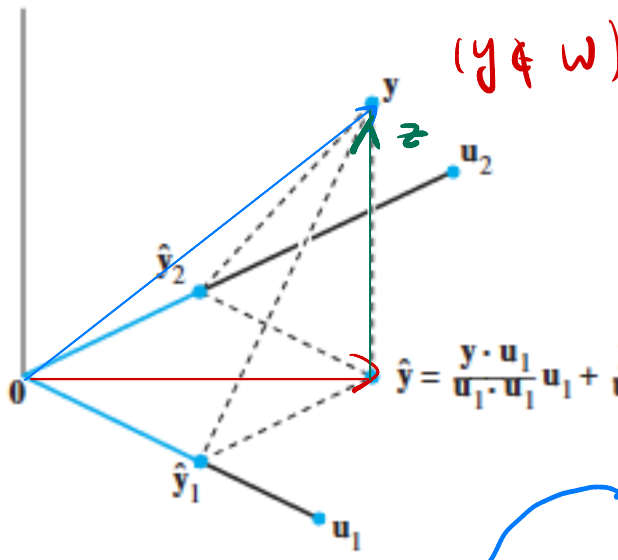
$$= \frac{3}{10}\begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2/5 \\ 2 \\ 1/5 \end{pmatrix}$$

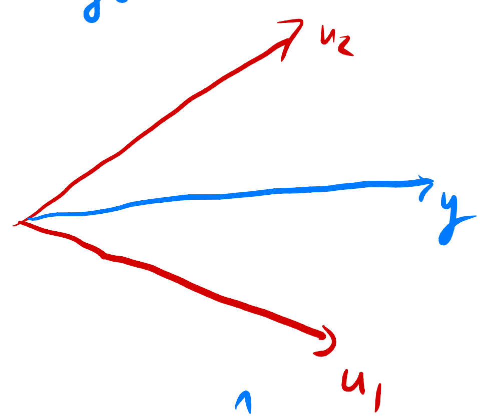
$$z = y - \hat{y} = \begin{pmatrix} 7/5 \\ 0 \\ 14/5 \end{pmatrix}$$

A Geometric Interpretation of the Orthogonal Projection:

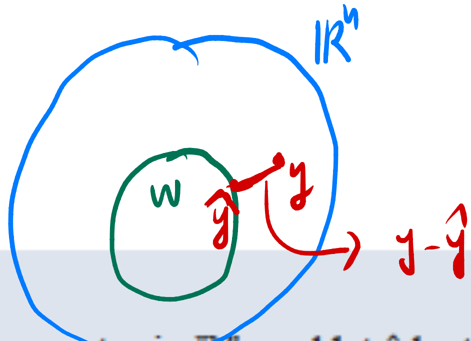
$y \in W$



$$\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \hat{\mathbf{y}}_1 + \hat{\mathbf{y}}_2$$



$y = \hat{y}$



Theorem:

The Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, let $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and let $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the closest point in W to $\mathbf{y}$, in the sense that

$$\|\mathbf{y} - \hat{\mathbf{y}}\| < \|\mathbf{y} - \mathbf{v}\| \tag{3}$$

for all $\mathbf{v}$ in W distinct from $\hat{\mathbf{y}}$.

def: distance between y & subspace W.

: $\|y - \hat{y}\|$

The vector $\hat{\mathbf{y}}$ is called the best approximation to $\mathbf{y}$ by elements of W.

Example 2: Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$ in Example 1. Determine the distance from $\mathbf{y}$ to. the subspace W

To find $\hat{y}$.

$\hat{y} = \begin{pmatrix} -2/5 \\ 2 \\ 1/5 \end{pmatrix}$ is the closed point to y in W.

distance $\|y - \hat{y}\| = \|z\| = \left\| \begin{pmatrix} 7/5 \\ 0 \\ 14/5 \end{pmatrix} \right\| = \frac{1}{5}\sqrt{49 + 14^2}$

Example 3: Find the best approximation to $\mathbf{z}$ by vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

$$\mathbf{z} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}$$

$\hookrightarrow$ span$\{v_1\ v_2\} := W$

always check: v_1 & v_2 are orthogonal to each other.

by the "Best approximation theorem", we just need to find $\hat{y}$

$$\hat{y} = \frac{v_1 \cdot z}{v_1 \cdot v_1} v_1 + \frac{v_2 \cdot z}{v_2 \cdot v_2} v_2 = \frac{4+0+0+3}{4+1+9}\begin{pmatrix} 2 \\ 0 \\ -1 \\ -3 \end{pmatrix} + \frac{10-8+0-2}{25+4+16+4}\begin{pmatrix} 5 \\ -2 \\ 4 \\ 2 \end{pmatrix}$$

Definition: An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem: If $\{\mathbf{u}_1, \cdots, \mathbf{u}_p\}$ is $\boxed{\text{an orthonormal basis}}$ for a subspace W of $\mathbb{R}^n$, then

$$\hat{y} = \text{proj}_W \mathbf{y} = (\mathbf{y}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{y}\cdot\mathbf{u}_2)\mathbf{u}_2 + \cdots + (\mathbf{y}\cdot\mathbf{u}_p)\mathbf{u}_p$$

If $U = [\mathbf{u}_1\ \mathbf{u}_2\ \cdots\ \mathbf{u}_p]$, then

this in 6.2

$$U^tUy = Iy = y$$

$$\text{proj}_W \mathbf{y} = UU^T\mathbf{y} \quad \text{for all } \mathbf{y} \text{ in } \mathbb{R}^n$$

Example 4: Let $\mathbf{y} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1\}$.

$\|\vec{u}_1\| = \sqrt{\frac{1}{10} + \frac{9}{10}} = 1$

(1) . Let U be the 2×1 matrix whose only column is $\mathbf{u}_1$. Compute U^TU and UU^T .

(2) . Compute $\text{proj}_W \mathbf{y}$ and $(UU^T)\mathbf{y}$.

(1) $\underset{1\cdot 2}{(U^t)}\ U_{2\cdot 1} = I = 1.$

UU^t is always a symmetric matrix. $(A = A^t)$

$$\underset{2\cdot 1}{U}\ \underset{1\cdot 2}{U^t} = \begin{pmatrix} \frac{1}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{9}{10} \end{pmatrix}$$

(2) $$\text{Proj}_W y = UU^t\vec{y} = \begin{pmatrix} \frac{1}{10} & \frac{-3}{10} \\ \frac{-3}{10} & \frac{9}{10} \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 9 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ 6 \end{pmatrix}$$