Section 6.4 The Gram-Schmidt Process (GS)

The Gram–Schmidt process is a simple algorithm for producing an orthogonal or orthonormal basis for any nonzero subspace of $\mathbb{R}^n$.

Theorem:

The Gram–Schmidt Process

Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ (linearly indep.) for a nonzero subspace W of $\mathbb{R}^n$, define

$$\mathbf{v}_1 = \mathbf{x}_1$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \boxed{\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1}$$

$\hat{y}$

vector produced by subtracting from x_2 its projection onto $W_1 = \text{span}\{v_1\}$

z

$$\mathbf{v}_3 = \mathbf{x}_3 - \underbrace{\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2}_{\hat{y} = \text{proj}_{W_2} x_3}$$

→ vector produced by subtracting from x_3 its projection onto W_2

$\hat{y} = \text{proj}_{W_2} x_3$, $W_2 = \text{span}\{v_1 \& v_2\}$

$$\vdots$$

$$\mathbf{v}_p = \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ (output) is ① an orthogonal basis for W. In addition

② $$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \quad \text{for } 1 \le k \le p \tag{1}$$

Orthonormal bases: An orthonormal basis is constructed easily from an orthogonal basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$: simply normalize (i.e., "scale") all the $\mathbf{v}_k$. → $\frac{v_1}{\|v_1\|}$ $\frac{v_2}{\|v_2\|}$ ... $\frac{v_p}{\|v_p\|}$

Example 1: Given a basis for a subspace W:

$$\begin{array}{cc} x_1 & x_2 \\ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, & \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} \end{array}$$

Construct an orthogonal basis for W.

$$v_1 = x_1 = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= \begin{pmatrix} 5 \\ 6 \\ -7 \end{pmatrix} - \frac{0+24-14}{0+16+4}\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -8 \end{pmatrix}$$

orthonormal basis

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} \Big/ \sqrt{0+16+4}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} 5 \\ 4 \\ -8 \end{pmatrix} \Big/ \sqrt{25+16+64}$$

$\{u_1, u_2\}$ is an orthonormal basis.

Example 2: Let

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Then $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is clearly linearly independent and thus is a basis for a subspace W of $\mathbb{R}^4$. Construct an orthogonal basis for W.

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{1+1+1}{1+1+1+1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}$$

Suppose $p \cdot q = 0$ $\quad c \in \mathbb{R}$.

$$p \cdot (cq) = c\,(p \cdot q) = 0$$

$$v_2' = 4 \cdot v_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (v_2' \cdot v_1 \text{ still } = 0)$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2'}{v_2' \cdot v_2'} v_2'$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1+1}{1+1+1+1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{9+1+1+1} \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$\{ v_1, v_2', v_3' \}$$

$A = PDP^{-1}$

The QR Factorization: If A is an $\boxed{m \times n \text{ matrix}}$ with $\boxed{\text{linearly independent columns,}}$ then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

1 2 3

Q : GS (cols of A) $\xrightarrow{\text{then}}$ normalize it.

R : $A = QR$

$$Q^T A = \underbrace{Q^T Q}_{I} R = I \cdot R = R.$$

(thm in 6-2)

Example 3: Find the QR Factorization of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$$

$$u_1 = \frac{v_1}{\|v_1\|} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Big/ \sqrt{1+1+1+1}$$

$$u_2 = \frac{v_2}{\|v_2\|} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Big/ \sqrt{9+1+1+1}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \begin{pmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{pmatrix} \Big/ \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}}$$

$\{u_1, u_2, u_3\}$ is an orthonormal basis

The QR Factorization: If A is an $m \times n$ matrix with linearly independent columns, then A can be factored as $A = QR$, where Q is an $m \times n$ matrix whose columns form an orthonormal basis for Col A and R is an $n \times n$ upper triangular invertible matrix with positive entries on its diagonal.

by QR Factorization

$$Q = [u_1 \ u_2 \ u_3]$$

$$R = Q^t A = \begin{pmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{pmatrix}$$

Example 3: Find the QR Factorization of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$