Section 6.7 Inner Product Spaces

Definition: An inner product on a vector space V is a function that, to each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V, associates a real number $\langle \mathbf{u}, \mathbf{v} \rangle$ and satisfies the following axioms, for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and all scalars c:

Dot product of $u \in \mathbb{R}^n$ & $v \in \mathbb{R}^n$ is an (one) inner product of $\mathbb{R}^n$.

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{u}+\mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3. $\langle c\mathbf{u}, \mathbf{v} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle$ $\quad c \in \mathbb{R}$
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ and $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

A vector space with an inner product is called an inner product space.

Definition: The length or norm of a vector $\mathbf{v}$ is

$$||\mathbf{v}|| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

dot product $\|u\| = \sqrt{u \cdot u}$

A unit vector is one whose length is 1.

The distance between $\mathbf{u}$ and $\mathbf{v}$ is $||\mathbf{u}-\mathbf{v}||$.

Vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.

Example 1: Let set of real numbers $\mathbb{R}^2$ have the inner product

$$\langle \mathbf{u}, \mathbf{v} \rangle = 4u_1v_1 + 4u_2v_2.$$

dot product. $u \cdot v = u_1v_1 + u_2v_2$

Let $\mathbf{x} = (3,2)$ and $\mathbf{y} = (4,-1)$. Compute $\langle \mathbf{x}, \mathbf{y} \rangle$, $||\mathbf{x}||$, and $||\mathbf{y}||$.

$$\langle x, y \rangle = 4 \cdot \underset{u_1}{3} \cdot \underset{v_1}{4} + 4 \cdot \underset{u_2}{2}\,\underset{v_2}{(-1)} = 40$$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{4 \cdot 3 \cdot 3 + 4 \cdot 2 \cdot 2} = \sqrt{52}$$

$$\|y\| = \sqrt{\langle y, y \rangle} = \sqrt{4 \cdot 4 \cdot 4 + 4(-1)(-1)} = \sqrt{68}$$

$\boxed{\mathbb{P}_2 \text{ has an inner product given by evaluation at } -1, 0, 1}$

Example 2: Let $\mathbb{P}_2$ have the inner product

$\langle\{1, t, t^2\}\rangle$

$$\langle p, q \rangle = p(t_0)q(t_0) + p(t_1)q(t_1) + p(t_2)q(t_2)$$

for any p and q in $\mathbb{P}_2$. And let $t_0 = -1$, $t_1 = 0$, and $t_2 = 1$.
Compute the lengths of the vectors $p(t) = 2t^2 + 1$ and $q(t) = 3t + 5$. Compute the inner product $\langle p, q \rangle$.

$$\|p\| = \sqrt{\langle p, p\rangle} = \sqrt{p(-1)\,p(-1) + p(0)\,p(0) + p(1)\,p(1)}$$

$$= \sqrt{(2+1)(2+1) + 1\cdot 1 + (2+1)(2+1)} = \sqrt{19}$$

$$\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{q(-1)\,q(-1) + q(0)\,q(0) + q(1)\,q(1)}$$

$$= \sqrt{(-3+5)(-3+5) + 5\cdot 5 + (3+5)(3+5)} = \sqrt{4+25+64}$$

$$\langle p, q\rangle = p(-1)\,q(-1) + p(0)\,q(0) + p(1)\,q(1)$$

$$= (2+1)(-3+5) + 1\cdot 5 + (2+1)(3+5)$$

$$= 6 + 5 + 24 = 35.$$

The Gram–Schmidt Process:

The existence of orthogonal bases for finite-dimensional subspaces of an inner product space can be established by the Gram–Schmidt process, just as in $\mathbb{R}^n$.

$\langle p, q\rangle = p(-3)q(-3) + p(-1)q(-1) + p(1)q(1) + p(3)q(3)$

Example 3: Let $\mathbb{P}_3$ have the inner product given by evaluation at -3, -1, 1, and 3. Let $p_0(t) = 1$, $p_1(t) = t$, and $p_2(t) = t^2$.

(1). Compute the orthogonal projection of p_2 onto the subspace spanned by p_0 and p_1.

need to check $\langle p_0, p_1\rangle = 0$ (p_0 is orthogonal to p_1)

$$\langle p_0, p_1\rangle = p_0(-3)p_1(-3) + p_0(-1)\cdot p_1(-1) + p_0(1)p_1(1) + p_0(3)p_1(3)$$

$$= 1\cdot -3 + 1\cdot -1 + 1\cdot 1 + 1\cdot -3 = 0 \iff p_0 \perp p_1$$

the projection of p_2 onto span$\{p_0\ p_1\} := W$

(6.3)

$$\hat{p} = \frac{\langle p_2, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \frac{\langle p_2, p_1\rangle}{\langle p_1, p_1\rangle} p_1$$

(2). Find a polynomial q that is orthogonal to p_0 and p_1, such that $\{p_0, p_1, q\}$ is an orthogonal basis for Span $\{p_0, p_1, p_2\}$. Scale the polynomial q so that its vector of values at $(-3,-1,1,3)$ is $(1,-1,-1,1)$.

$$\langle p_2, p_0\rangle = p_2(-3)p_0(-3) + p_2(-1)p_0(-1) + p_2(1)p_0(1) + p_2(3)p_0(3)$$

$p_2(t) = t^2$ $p_0(t) = 1$

$$= 9\cdot 1 + 1\cdot 1 + 1\cdot 1 + 9\cdot 1 = 20$$

$$\langle p_0, p_0\rangle = p_0(-3)p_0(-3) + \cdots + p_0(3)p_0(3)$$

$$= 4$$

$$\langle p_2, p_1\rangle = p_2(-3)p_1(-3) + p_2(-1)p_1(-1) + p_2(1)p_1(1) + p_2(3)p_1(3)$$

$$= -9\cdot 3 - 1\cdot 1 + 1\cdot 1 + 9\cdot 3 = 0$$

$\hat{p} = 5 \cdot p_0 = 5$ (orthogonal projection of p_2 onto span$\{p_0\ p_1\}$).

(2) $q \perp$ span$\{p_0\ p_1\}$

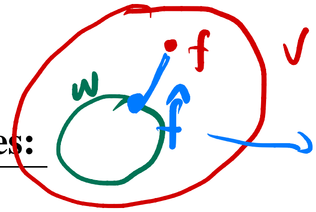
$\underbrace{q = p - \hat{p}}_{\text{orthogonal decomposition.}}$ $\hat{p} \in$ span$\{p_0\ p_1\}$

$q = t^2 - 5$

the vector of values of q at $(-3, -1, 1, 3)$: $\begin{pmatrix} q(-3) \\ q(-1) \\ q(1) \\ q(3) \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ -4 \\ 4 \end{pmatrix}$

we just need to scale q by $\frac{1}{4}$

$\hat{q} = \frac{1}{4}(t^2 - 5)$



Best Approximation in Inner Product Spaces:

$\hat{f}$ is just the orthogonal projection of f onto W.

One problem is to approximate a function f in V by a function g from a specified subspace W of V. The "closeness" of the approximation of f depends on the way $||f-g||$ is defined. We will consider only the case in which the distance between f and g is determined by an inner product. In this case, the best approximation to f by functions in W is the orthogonal projection of f onto the subspace W.

formula check (6.3)

Example 4: Let $\mathbb{P}_3$ have the inner product as in Example 3, with $p_0(t)$, $p_1(t)$ and q the polynomials described there. Find the best approximation to $p(t)=t^3$ by polynomials in Span $\{p_0, p_1, q\}$.

$\Leftrightarrow$ Find a vector $\hat{p}$ in span$\{p_0\ p_1\ q\}$ such that

$$\|\hat{p}-p\| \le \|y-p\| \quad \text{for all } y \in \text{span}\{p_0\ p_1\ q\}$$

by the best approximation thm (6.3)

$$\hat{p} = \text{orthogonal projection of } p(t)=t^3 \text{ onto span}\{p_0\ p_1\ q\}$$

$$\hat{p} = \frac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle}\underbrace{p_0}_{1} + \frac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle}\underbrace{p_1}_{t} + \frac{\langle p, q\rangle}{\langle q, q\rangle}\underbrace{q}_{} = \frac{41}{5}t$$

$q = \frac{1}{4}(t^2-5)$

The Cauchy-Schwarz Inequality: For all $\mathbf{u}$ and $\mathbf{v}$ in V,

$$|\underbrace{\langle \mathbf{u}, \mathbf{v}\rangle}_{\text{scalar}}| \le ||\mathbf{u}||\,||\mathbf{v}||$$

absolute value

The Triangle Inequality: For all $\mathbf{u}$ and $\mathbf{v}$ in V,

$$||\mathbf{u}+\mathbf{v}|| \le ||\mathbf{u}|| + ||\mathbf{v}||$$