Section 7.1 symetric matrix : $A = A^t$ (symmetric across the diag A)

↳ Square matrix

THEOREM 1 If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

$\lambda_1 \leftrightarrow v_1$
$\neq$
$\lambda_2 \leftrightarrow v_2$
$\Rightarrow$ v_1 is orthogonal to v_2.

THEOREM 2 An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix.

def: A is called orthogonally diagonalizable if there exist an orthogonal matrix $(P^t = P^{-1})$ P & a diagonal matrix D such that $A = PDP^{-1}$

THEOREM 3 **The Spectral Theorem for Symmetric Matrices**

An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

thm 1 c. The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

thm 2 d. A is orthogonally diagonalizable.

Spectral Decomposition

Suppose $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ of A and the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are in the diagonal matrix D. Then, since $P^{-1} = P^T$,

$$
\begin{aligned}
A = PDP^T &= \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 \mathbf{u}_1 & \cdots & \lambda_n \mathbf{u}_n \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}
\end{aligned}
$$

Using the column–row expansion of a product (Theorem 10 in Section 2.4), we can write

$$
A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T \tag{2}
$$