MA26500 FINAL EXAM INSTRUCTIONS Spring 2019

GREEN - Test Version 01

NAME ______________________________ INSTRUCTOR ______________________________

1. You must use a **#2 pencil** on the mark–sense sheet (answer sheet).
2. On the mark–sense sheet, fill in the **instructor's** name (if you do not know, write down the class meeting time and location) and the **course number** which is **MA265**.
3. Fill in your **NAME** and blacken in the appropriate spaces.
4. Fill in the **SECTION Number** boxes with section number of your class (see below if you are not sure) and blacken in the appropriate spaces.

172	2:30pm	MWF	Brown, Johnny	173	10:30am	MWF	Chen, Ying
174	10:30am	TR	Ho, Meng-Che	175	12:00pm	TR	Ho, Meng-Che
176	10:30am	TR	Liu, Baiying	177	4:30pm	TR	Liu, Baiying
178	1:30pm	MWF	Liu, Tong	179	10:30am	MWF	Liu, Tong
180	1:30pm	TR	Luo, Tao	181	12:00pm	TR	Luo, Tao
182	4:30pm	TR	Madsen, Caroline	183	3:00pm	TR	Madsen, Caroline
184	12:30pm	MWF	Moon, Yong Suk	185	11:30am	MWF	Moon, Yong Suk
186	3:30pm	MWF	Patzt, Peter	187	4:30pm	MWF	Patzt, Peter
188	10:30am	MWF	Wang, Xu	189	11:30am	MWF	Wang, Xu
190	9:30am	MWF	Wang, Yating	191	8:30am	MWF	Wang, Yating
192	1:30pm	MWF	Wei, Ning	193	3:30pm	MWF	Wei, Ning
194	9:30am	MWF	Xu, Ping	195	10:30am	MWF	Xu, Ping
196	1:30pm	TR	Yang, Zhiguo	197	3:00pm	TR	Yang, Zhiguo

5. Fill in the correct TEST/QUIZ NUMBER (**GREEN** is 01).
6. Fill in the 10-DIGIT PURDUE ID and blacken in the appropriate spaces.
7. Sign the mark–sense sheet.
8. Fill in your name and your instructor's name on the question sheets (above).
9. There are 25 questions, each worth 8 points. Blacken in your choice of the correct answer in the spaces provided for questions 1–25 in the answer sheet. Do all your work on the question sheets, in addition, also **CIRCLE** your answer choice for each problem on the question sheets in case your scantron is lost. **Turn in both the mark-sense sheets and the question sheets to your instructor when you are finished**.
10. Show your work on the question sheets. Although no partial credit will be given, any disputes about grades or grading will be settled by examining your written work on the question sheets.
11. NO CALCULATORS, BOOKS, NOTES, PHONES OR CAMERAS ARE ALLOWED on this exam. Turn off or put away all electronic devices. Use the back of the test pages for scrap paper.

1. The matrix below represents the augmented matrix of a system of linear equations. Assume that the variables in this system are x_1, x_2, x_3, x_4, x_5, and x_6, and let A be the coefficient matrix:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 1 & d \end{array}\right)$$

Which of the following statements are **true**?

(i) For any given c and d, the system above is consistent.

(ii) The coefficient matrix A is in reduced echelon form.

(iii) The right hand side vector is in the column space of matrix A.

(iv) The system has no solution.

(v) The system has infinitely many solutions.

A. (i), (ii) only

B. (i), (ii), (iii) only

C. (ii), (iii), (iv) only

D. (i), (iii), (v) only

E. all of the above

2. Suppose the set $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ a \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right\}$ is linearly dependent. Find a.

A. $a = -5$.

B. $a = -2$.

C. $a = 2$.

D. $a = 1$.

E. $a = -3$.

3. Which of the following statements is **false**?

A. If $\{v_1, v_2, v_3\}$ is linearly dependent, then v_3 is a linear combination of v_1 and v_2.

B. Suppose that the columns of A are v_1, v_2, and v_3. Then the matrix equation $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$ is equivalent to the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = b$.

C. Suppose that X_0 is a solution to the linear system $AX = b$. Then $\{X|AX = b\} = X_0 + \{X|AX = 0\}$.

D. The columns of A are linearly independent if and only if A has a pivot position in every column.

E. A homogeneous linear system has a non-trivial solution if and only if it has at least one free variable.

4. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that $L(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$ and $L(\begin{bmatrix} 3 \\ 2 \end{bmatrix}) = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Find $L(\begin{bmatrix} -1 \\ 2 \end{bmatrix})$.

A. $\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

B. $\begin{bmatrix} 2 \\ 2 \\ 7 \end{bmatrix}$

C. $\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ 6 \\ -3 \end{bmatrix}$

E. $\begin{bmatrix} 5 \\ 3 \\ 7 \end{bmatrix}$

5. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation whose standard matrix is $\begin{bmatrix} t-1 & 2t-2 \\ 1 & t \end{bmatrix}$ where t is a real number. Find ALL values of t such that L is one-to-one.

A. $t \neq 1$

B. $t \neq 0, 1$

C. $t \neq 1, 2$

D. $t = 1$

E. $t = 2$

6. Let $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 0 & 2 \\ -3 & 1 & -3 \end{bmatrix}$ and let its inverse $A^{-1} = \begin{bmatrix} b_{ij} \end{bmatrix}$. Find the trace of the matrix A^{-1}. In other words, compute the sum $b_{11} + b_{22} + b_{33}$.

A. -1

B. 0

C. $\frac{1}{2}$

D. 1

E. 2

7. Find the third column of the matrix D, given that $C = \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}$ and $CD = \begin{bmatrix} 2 & 0 & -2 & 1 & 0 \\ 0 & 3 & 1 & 7 & 0 \end{bmatrix}$.

A. $\begin{bmatrix} -\frac{12}{5} \\ \\ \frac{7}{5} \end{bmatrix}$

B. $\begin{bmatrix} -\frac{3}{5} \\ \\ \frac{7}{5} \end{bmatrix}$

C. $\begin{bmatrix} -\frac{2}{5} \\ \\ \frac{1}{5} \end{bmatrix}$

D. $\begin{bmatrix} -5 \\ -7 \end{bmatrix}$

E. $\begin{bmatrix} -7 \\ -5 \end{bmatrix}$

8. Let A be an $m \times n$ matrix. Which of the following statement is necessarily **true**?

A. The nullity of A is the same as the nullity of A^T.

B. The rank of A is the same as the rank of A^T.

C. The column space of A is the same as the null space of A^T.

D. The columns of A form a basis of the column space of A.

E. The columns of A^T form a basis of the null space of A.

9. Let

$$A = \begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 3 & 2 & 3 & 1 & 3 \\ 3 & 1 & 2 & 2 & 2 \end{bmatrix}$$

Which of the following is a basis of the null space of A?

A. $\begin{bmatrix} -4 \\ -8 \\ 9 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

B. $\begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$

C. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

D. $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

E. $\begin{bmatrix} 4 \\ 8 \\ -11 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -8 \\ -16 \\ 19 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -4 \\ -8 \\ 0 \\ 1 \\ 9 \end{bmatrix}$

10. Suppose a 3×3 matrix $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ has determinant 4. What is the determinant of

$$B = \begin{bmatrix} a & 2b & c \\ g & 2h & i \\ d+3a & 2e+6b & f+3c \end{bmatrix}?$$

A. 4.

B. 8.

C. -8.

D. 24.

E. -24.

11. Compute the value of the following determinant:

$$\begin{bmatrix} 4 & -9 & 2 & 3 \\ 0 & 3 & 0 & -4 \\ -5 & 0 & 0 & 3 \\ 0 & 5 & 0 & -7 \end{bmatrix}.$$

A. 10.

B. -10.

C. 410.

D. -410.

E. 90.

12. Suppose $A = PDP^{-1}$, where P is a 3×3 invertible matrix and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

Let $B = 2I + 3A + A^2$, which of the following is **true**?

A. B is not diagonalizable.

B. B is diagonalizable, and $B = PCP^{-1}$, where $C = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

C. B is diagonalizable, and $B = PCP^{-1}$, where $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

D. B is diagonalizable, and $B = PCP^{-1}$ for some C, but there is not enough information to determine C.

E. There is not enough information to determine whether B is diagonalizable.

$$B = 2I + 3A + A^2$$

$$I = PP^{-1} = P \cdot I P^{-1}$$

$$A = PDP^{-1}$$

$$A^2 = PD^2P^{-1}$$

$$A^k = PD\cancel{P^{-1}} \cdot \cancel{P}D\cancel{P^{-1}} \cdots \cancel{P}DP^{-1}$$

$$= P D^k P$$

$$D = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

$$D^k = \begin{pmatrix} a_1^k & & & \\ & a_2^k & & \\ & & \ddots & \\ & & & a_n^k \end{pmatrix}$$

$$B = 2PIP^{-1} + 3PDP^{-1} + PD^2P^{-1}$$

$$= P \cdot (2I) \cdot P^{-1} + P(3D)P^{-1} + PD^2P^{-1}$$

$$= P(\underbrace{2I + 3D + D^2}_{\text{diagonal matrix}}) \cdot P^{-1}$$

$\Rightarrow$ B is diagonalizable.

$$C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

R1. $A \sim B$, $\Rightarrow$ they have the same characteristic eqn.

$\not\Leftarrow$ not true.

R2. $A \sim B$, A is diagonalizable $\Rightarrow$ B is also diagonalizable.

If $A \sim B$, $A = PBP^{-1}$

A is diag, $A = R\boxed{D}R^{-1}$, D is a diag matrix.

$$PBP^{-1} = RDR^{-1}$$

$$PB\cancel{P^{-1}P} = RDR^{-1} \cdot P$$

$$\cancel{P^{-1}P}B = P^{-1}RDR^{-1}P$$

$$B = P^{-1}R\boxed{D}R^{-1}P.$$

need to show $P^{-1}R$ is invertible.

but $\det(P^{-1}R) = \underbrace{\det(P^{-1})}_{\neq 0}\underbrace{\det(R)}_{\neq 0} \neq 0$

$\Rightarrow$ $P^{-1}R$ is invertible $\Rightarrow$ B is diagonalizable. //

$Av = \lambda v$, $\boxed{v \neq 0}$

13. Which of the following statements are **true**?

$(-A)\cdot v = (-\lambda)\cdot v$

(i) If λ is an eigenvalue for A, then $-\lambda$ is an eigenvalue for $-A$.

(ii) If $\underbrace{\text{zero is an eigenvalue of } A}_{S_1}$, then $\underbrace{A \text{ is not invertible}}_{S_2}$.

check S_3.

(iii) If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.

↳ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(iv) Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then A is both invertible and diagonalizable.

A. (i) and (ii) only
B. (i) and (iii) only
C. (i), (ii) and (iii) only
D. (i), (ii) and(iv) only
E. (i), (ii), (iii) and (iv)

(ii). If 0 is an eig val of A.

(I) $A\cdot v = 0 \cdot v = 0$, $(v \neq 0)$

$\Rightarrow \dim(\text{null}(A)) \geq 1$.

$\Rightarrow \text{Rank}(A) + \dim(\text{null}(A)) = \underset{\#\text{col of } A}{n}$.

$\Rightarrow \underset{\dim(\text{col}(A))}{\text{Rank}(A)} < n \Rightarrow A$ is not invertible.

(II) $\det(A - \lambda I) = 0$

$\det(A) = 0 \Rightarrow A$ is singular.

$S_1 \Rightarrow S_2$, $\xrightarrow{\text{math logic.}}$ $\bar{S_2} \Rightarrow \bar{S_1}$

$\Rightarrow$ If A is invertible

$\Rightarrow$ 0 is not an eigvalue of A.

14. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x) = Ax$, where $A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$. Which of the following is a basis $\mathcal{B}$ for $\mathbb{R}^2$ with the property that the $\mathcal{B}$-matrix for T is a diagonal matrix?

A. $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$.

(iv) $A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

$\lambda_1 = \lambda_2 = 2$

Find the eigenvectors,

$$(A - \lambda I) \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

eig (λ)
= null ($A - \lambda I$)

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$V = s \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underline{s \in \mathbb{R}} \Rightarrow \text{eig space.}$$

$\Rightarrow$ dim (eig (λ)) = 1. $\Rightarrow$ not diagonalizable.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda_1 = \lambda_2 = 1$.

$(A - \lambda I) V = 0 \iff \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0$

$\Rightarrow$ a & b are both free

$\Rightarrow$ null $(A - \lambda I)$ = eig (λ) = $s \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, s, t free

Diagonalization of A,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

(i) λ is an eigenvalue of A, λ^2 is an eigval of A^2.

$\Downarrow$

Pf. $\det(A - \lambda I) = 0$

need to show $\det(A^2 - \lambda^2 I) = 0$

but $\det(A^2 - \lambda^2 I) = \det(A^2 - \lambda^2 I^2)$

$a^2 - b^2 = (a-b)(a+b)$

$A^2 - B^2 \overset{\times}{=} (A+B)(A-B)$

$\Downarrow$

$AB \neq BA$.

$= \det((A - \lambda I) \cdot (A + \lambda I))$

$= \det(A - \lambda I) \cdot \det(A + \lambda I)$

$\underset{0}{\shortparallel}$

$= 0$

15. Let $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, where $i = \sqrt{-1}$. Then A^{32} equals

A. $\begin{bmatrix} 1 & i \\ 1 & i \end{bmatrix}$

B. $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

C. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

D. $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

E. $\begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}$.

16. Consider the dynamical system $x' = Ax$, where $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. Then the origin is

A. an attractor

B. a repeller

C. a saddle point

D. a spiral point

E. none of the above

$Av = \lambda v$ $(\lambda \in \mathbb{R})$

$\det(A - \lambda I) = 0.$

$(A - \lambda I)v = 0$ (eig-vector $\neq 0$)

$\det\left(\begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$

$\det\begin{pmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{pmatrix} = 0$

$\lambda(\lambda - 1) - 2) = 0$

$\lambda^2 - \lambda - 2 = 0$

$\Rightarrow (\lambda - 2)(\lambda + 1) = 0$

$\Rightarrow \lambda_1 = 2$ & $\lambda_2 = -1$

real e-vals with opposite signs $\Rightarrow$ origin is a saddle point.

17. Which one of the following is the solution to the differential equation

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

with initial condition $\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$?

A. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3e^{4t} \\ 3e^{4t} \end{bmatrix} - \begin{bmatrix} 2e^{-t} \\ 3e^{-t} \end{bmatrix}$

B. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 3e^{4t} \\ 3e^{4t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ -3e^{-t} \end{bmatrix}$

C. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2e^{4t} \\ 2e^{4t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ -3e^{-t} \end{bmatrix}$

D. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2e^{4t} \\ 2e^{4t} \end{bmatrix} - \begin{bmatrix} 3e^{-t} \\ -2e^{-t} \end{bmatrix}$

E. $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} 2e^{4t} \\ 2e^{4t} \end{bmatrix} + \begin{bmatrix} 3e^{-t} \\ -2e^{-t} \end{bmatrix}$

$$A = \begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$x'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = A \cdot x \quad (*)$$

$\lambda_1 \quad \lambda_2$ 2 distinct real eigenvalues

$v_1 \quad v_2$ eigenvectors.

general solution of $(*)$.

$$c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}, \quad c_1 \,\&\, c_2 \text{ are 2 consts.}$$

$$\lambda_1 = -1 \quad v_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\lambda_2 = 4 \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

general solution.

$$c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

I.V.P $\Rightarrow$ you need to find c_1 & c_2.

$t=0$

$$\begin{array}{r} x_1(0) \\ x_2(0) \end{array} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-0} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4\cdot 0}$$

$$\begin{pmatrix} 5 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 = -1 \\ c_2 = 3. \end{cases}$$

$\Rightarrow$ solution to IVP.

$$-1 \begin{pmatrix} -2 \\ 3 \end{pmatrix} e^{-t} + 3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{4t}$$

18. Which of the following subsets of the vector space $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$?

(i) The set of all vectors $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with the property $2xyz = 0$.

(ii) The set of all the solutions of the equation $x - 5y + 2z = 0$.

(iii) The set of all solutions for the system $\begin{bmatrix} 2 & 3 & 2 \\ 5 & 2 & 8 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(iv) The set of all the solutions of the equation $x + 3y = 2z + 1$.

A. (ii) and (iii) only

B. (ii) and (iv) only

C. (iii) and (iv) only

D. (ii), (iii) and (iv) only

E. (i), (ii), (iii) and (iv)

19. Determine a basis for the set spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}, v_3 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, v_4 = \begin{bmatrix} 5 \\ 11 \\ 17 \end{bmatrix}, v_5 = \begin{bmatrix} 2 \\ 7 \\ 12 \end{bmatrix}, v_6 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

A. $\{v_1, v_3, v_4\}$

B. $\{v_1, v_3, v_5\}$

C. $\{v_2, v_3, v_4\}$

D. $\{v_3, v_4, v_5\}$

E. $\{v_1, v_3, v_6\}$

20. Performing the Gram-Schmidt process on the vectors $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} \right\}$ yields an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ of $\mathbb{R}^3$. What is $\mathbf{u}_3$?

A. $\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

B. $\frac{1}{\sqrt{26}} \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix}$

C. $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

D. $\frac{1}{\sqrt{14}} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

E. $\frac{1}{\sqrt{17}} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$

21. Find the least squares solution to

$$\begin{bmatrix} 1 & 3 \\ 2 & 2 \\ 1 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix}.$$

A. $(0, 1)$
B. $(1, 1)$
C. $(1, 2)$
D. $(0, 2)$
E. $(2, 1)$

22. Find the distance from the vector $\mathbf{y}$ to the subspace $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

A. 12.

B. $2\sqrt{2}$.

C. $3\sqrt{3}$.

D. 8.

E. $3\sqrt{5}$.

23. Let A be an $n \times n$ matrix. Which of the following statements is/are **NOT** equivalent to that A is invertible?

(i) Columns of A are linearly independent.

(ii) A is diagonalizable.

(iii) Columns of A is an orthonormal set.

(iv) The dimension of the null space of A is 0.

(v) The linear system $AX = b$ always has solution for any $b \in \mathbb{R}^n$.

A. (i), (ii) and (iii) only.

B. (i) and (ii) only.

C. (ii) and (iii) only.

D. (i) and (iv) only.

E. (ii), (iv), (v) only.

24. Let $C[-1, 1]$ be the space of all continuous functions over $[-1, 1]$ with the inner product

$$\langle f(t), g(t) \rangle = \int_{-1}^{1} f(t)g(t)dt \quad \text{for any } f(t), g(t) \in C[-1, 1].$$

Which of the following set is an orthogonal basis of $\text{Span}\{1, t-1, t^2+t\}$?

A. $1,\ t,\ t^2$

B. $1, t-1, t^2+t$

C. $1, t, t^2-1$

D. $1, t-1, t^2$

E. $1, t, t^2 - \frac{1}{3}$.

25. Suppose that $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = QDQ^T$ where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and Q is an orthogonal matrix. In the following select a pair of Q and D with required properties.

A. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

B. $Q = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

C. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

D. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

E. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

HW Sec 4.2 Q42 (textbook)

Define $T: \mathbb{P}_2 \to \mathbb{R}_2$ by $T(p(t)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$

(a) Find $p(t) = 3 + 5t + 7t^2$, find $T(p(t))$

(b) Find a polynomial p in $\mathbb{P}_2$ that spans $\ker(T)$.

solution. $\mathbb{P}_2$: std basis, $1, t, t^2$.

$$p(t) = a + bt + ct^2.$$

$$T(p(t)) = T(\underbrace{3 + 5t + 7t^2})$$

$$= \begin{array}{cc} \begin{pmatrix} 3 + 5 \cdot 0 + 7 \cdot 0 \\ 3 + 5 \cdot 1 + 7 \cdot 1 \end{pmatrix} & \begin{array}{l} p(0) \\ p(1) \end{array} \end{array}$$

$$= \begin{pmatrix} 3 \\ 15 \end{pmatrix}$$

$$T(a+bt+ct^2) = \begin{pmatrix} a \\ a+b+c \end{pmatrix}$$

when $t=0$

$t=1$.

(b) $\ker(T) = \{ x$ in the domain of T,

such that $Tx = 0 \}$

we need to find $p(t) = a+bt+ct^2$,

need to find all $a\ b\ c$ such that

$$T(a+bt+ct^2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ a+b+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \left\{ s\begin{pmatrix}0\\-1\\1\end{pmatrix}, \ s\in\mathbb{R}\right\}$$

when $s=1$, $\Rightarrow$ $\begin{array}{c} \begin{pmatrix}0\\-1\\1\end{pmatrix} \end{array} \begin{array}{l} \to a \\ \to b \\ \to c \end{array}$

$$\Rightarrow p(t) = -t + t^2.$$