Brieskorn's theorem on Fundamental Group. Yilong Zhang
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Let $G$ be a simple Lie algebra over $\mathbb{C}$, $\mathfrak{g} \subset \mathfrak{g}$ Cartan subalgebra, $\mathfrak{h} \subset \mathfrak{g}$ the complexification and $W$ the Weyl group, with action on $\mathfrak{g}$ (and hence to $\mathfrak{h} \cap \mathfrak{g}$) generated by reflections. Remove the hyperplanes $H_{\lambda, \alpha} \cap \mathfrak{h}$ fixed by reflections, and denote it as $\mathfrak{h}^{\text{reg}}$. Therefore, the Weyl group acts on it freely. We will prove:

**Theorem (Brieskorn, 1971)** $\pi_1(\mathfrak{h}^{\text{reg}}/W, x) \cong B_w$

Here $B_w$ is the braid group associated to Weyl group $W$. Defined as $B_w = \langle T_i, i \in I \mid T_iT_jT_i^{-1} = T_jT_iT_j^{-1} \rangle$ by fixing a Weyl chamber with $I$ the index set of walls of the chamber.

**Example:** $G = SU_n$, $W = S_n$. $B_n$ is just the Artin Braid group.

$\langle T_1, \ldots, T_{n-1} \mid T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, [T_i, T_{i+1}] = 0, 1 \leq i < n - 1 \rangle$

With strands presentation:

\[
\begin{align*}
\begin{array}{cccc}
& & & \\
& & X & \\
& X & & \\
& & & X
\end{array}
\end{align*}
\]

Proof of theorem for $G = SU_n$: This is known by Fox-Neuwirth in 1962 that Artin's Braid group is isomorphic to the configuration space of unordered distinct points on $C_n$.

\[
\begin{array}{cc}
\mathbb{C}^n \setminus \{y_1, \ldots, y_n\} / \mathbb{Z}^n \cong C_n
\end{array}
\]

\[
\begin{array}{cc}
\pi_1(\mathfrak{h}^{\text{reg}}/W, x) \cong B_n
\end{array}
\]
Now, $\mathfrak{h}^{\text{reg}} = \{ (z_i, \ldots, z_n) \mid \sum_{i=1}^n z_i = 0, z_i \neq 0, \text{for some } i \}$ is contained in $Y_n$ as a hyperplane, intersecting by $\sum_{i=1}^n z_i = 0$, as it intersects $\{ z_i = 2j \}$ transversely for each $j$. We claim that $\mathfrak{h}^{\text{reg}} \times S^1 \cong Y_n$ via $(z_i, \ldots, z_n, t) \mapsto (2t z_i, \ldots, 2t z_n, t)$, shrinking the line to its origin gives a deformation retract from $Y_n$ to $\mathfrak{h}^{\text{reg}}$, which is equivariant under $S^1$ action, which shows $\tilde{T} C(\mathfrak{h}^{\text{reg}}, \sigma) \cong \tilde{T} C(Y_n, \sigma) = B_n$.

In general, we will first show the theorem for $\mathfrak{h} = \mathfrak{a}_2$ root space, and reduce the higher rank case to $\mathfrak{h} = \mathfrak{a}_2$ case.

**$\mathfrak{a}_2$** There are three such Lie algebras: $A_1$, $B_2$, $C_2$, with Weyl group $W = \langle s_1, s_2 \rangle$ ($s_i^2 = e, s_1 s_2 s_1 = e$) $m = 3, 4, 6$. (We can also include $m = 2$ for $A_1, A_2$.) Up to rotation of graph, the hyperplanes to be removed in $C^2$ are

![Graph with hyperplanes]

More precisely, choose $[1], [e]$, as standard basis on $C^2$, the action of $W$ is generated by rotation

\[
\begin{bmatrix}
\cos \frac{2\pi}{m} & -\sin \frac{2\pi}{m} \\
\sin \frac{2\pi}{m} & \cos \frac{2\pi}{m}
\end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\
0 & 1 
\end{bmatrix}
\]

reflections

Working over $\mathbb{C}$ we know we can diagonalize the rotation matrix, so by uniform $W$ action is given by

\[
\begin{bmatrix} e^{-2\pi i / m} & 0 \\
0 & e^{2\pi i / m}
\end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\
0 & 1 
\end{bmatrix}
\]
be, denote \( \sigma = e^{2\pi i \nu} \). Write \( C^2 \) by \( \sigma(u,v) \rightarrow (\sigma u, \sigma v) \)
\[
\sigma: (u,v) \rightarrow (v,u).
\]
As \( \sigma \trianglelefteq W \) is a normal subgroup. The action by \( W \) factors through a cyclic over \( C^2 \cong C^2/\langle \sigma \rangle \), we will first understand this space. Note that by the cyclic action \( \sigma: (u,v) \rightarrow (\sigma u, \sigma^m v) \), the invariant polynomials is an ideal generated by \( (uv, u^m, v^m) \). By setting \( x = uv, s = u^m, t = v^m \) we have relation \( x^m = st \), therefore \( C^2/\langle \sigma \rangle \) is identified with \( \mathbb{C}^2 \)
Now the induced reflection act on it by \( (x,s,t) \rightarrow (x,t,s) \). By change of variable \( s = y - 3, t = y + 2 \), we identify \( C^2/\langle \sigma \rangle \) by \( \{(x,y) \in \mathbb{C}^2 \mid x^m = y^2 \} \) with \( \mathbb{C}^2 \) and \( y \rightarrow -2 \) with branching locus \( \{x^m = y^2 \} \subset C^2 \). This shows

\[
\mathbb{C}^2/\langle \sigma \rangle \cong C^2 \setminus \{x^m = y^2 \}.
\]

So the problem reduces to calculate fundamental group of the complement of the plane curve \( y^2 = x^m \) in \( C^2 \).

Let \( C = \{y^2 = x^m \} \), the projection \( C^2 \setminus C \rightarrow C \) has fiber line minus \((x,y) \rightarrow x \)

no points everywhere except for at \( x = 0 \) denote to the fiber at \( x = 0 \). Then \( C^2 \setminus C \rightarrow C^* \) is a fiber bundle with fiber diffeomorphic to \( \mathbb{C} \setminus \{1, \pm \sqrt{m} \} \)

**Monodromy of fundamental group:** In general, let \( F \rightarrow E \rightarrow B \) be a fiber bundle.

that is, \( E \) and \( B \) are smooth manifolds, and for every point \( b \in B \), \( E_b \) is a fiber of \( E \) over \( b \).

such that \( E_b \cong \mathbb{C} \times F \). If we further assume that both \( F \) and \( B \) are path connected.

For \( b \in B \), choose \( \beta \in E_b \) that \( \beta \cdot B \subset F \) , then there is an action \( \pi_1(B, b) \cong \pi_1(B, \beta) \) (3)
Let assume there is a section \( s: B \to E \) (i.e. a smooth map \( s \circ \pi = \text{id} \)) and take \( \bar{e} = s(\bar{b}) \). Let \( \gamma: [0,1] \to B \) with \( \gamma(0) = \gamma(1) \) be a loop on \( B \) and \( \gamma: [0,1] \to F_{\bar{b}} \), with \( \gamma(0) = \gamma(1) = \bar{e} \) be a loop on \( F_{\bar{b}} \), we want to describe the action of \( \gamma \) on \( \bar{e} \).

Let \( \gamma^{*} E \to [0,1] \) defines a bundle over \([0,1]\), whose fiber \( \text{at} \ t \) is \( F_{\gamma(t)} \), with a preferred base point \( s(\gamma(t)) \), and the data \((F_{\gamma(t)}, s(\gamma(t)))\) varies continuously v.r.t \( t \). Now, we can deform \( \gamma \) into a loop \( \gamma' \) on \( F_{\bar{b}} \) based at \( s(\gamma(0)) \). As \( \gamma(0) = \gamma(1) = \bar{b} \), so \( s(\gamma(0)) = s(\gamma(1)) = \bar{b} \), we have \( \gamma' \) is a well defined loop in \( F_{\bar{b}} \) again based at \( \bar{b} \). so we define

\[
\pi_{1}(B, b) \to \pi_{1}(F_{\bar{b}}, \bar{b})
\]

\[(\gamma, \bar{e}) \mapsto \left[ \gamma^{*} \bar{e} \right] = : [\gamma] \cdot [\bar{e}]
\]

defined as above. By some lifting property, this is well-defined.

Now, go back to our case \( \pi: C^{x}|(C U \mathbb{L}o) \to C^{x} \) by shrinking the base to \( B = \{ x \in B \mid |x| < 2 \} \) and \( E := \pi^{-1}(B) \). We obtained our preferred bundle \( E \to B \) with a section \( s: B \to E \), \( \pi \to (x, N) \), where \( N \) is some large number on the real line such that \( x^{2} = x^{m} \) has no solution for \( x \in B \).

This is our preferred base point on each fiber.

Choose \( \gamma: [0,1] \to B \), \( t \to e^{2 \pi it} \), to be the loop around origin. we are keeping track of fibers \( F_{\exp(2\pi it)} \) along the unit circle. We found that for each \( k = m \) and on the interval \( \left[ \frac{k}{m}, \frac{k+1}{m} \right] \) as \( F_{\exp(2\pi i m k)} \) moves to \( F_{\exp(2\pi i m (k+1))} \) along the unit circle.
the "two holes" on the fiber are interchanged. In case $m = 3$, the loop $l_1$ keeps traces of "rotation" of the two holes.

In fact, this is exactly the monodromy action of $\gamma$ on $l_1$. If we define $l_{-1}$ to be the loop as follows:

Then it is not hard to see $[l_1]l_{-1}l_1 = [\beta^m l_1 \beta^m]$, where $[\beta] = l_1 l_{-1}$.

In general, we have:

**Proposition 2.** Let $E \to B$ be as above. and same for $\gamma$, $l_1$, $l_{-1}$, then

\[
\begin{align*}
[\beta] \cdot [l_1] & = \begin{cases} 
\beta^n l_{-1} \beta^n & m = 2n + 1 \\
\beta^{-n} l_1 \beta^n & m = 2n
\end{cases} \\
[\beta] \cdot [l_{-1}] & = \begin{cases} 
\beta^{-n} l_{-1} \beta^{-n} & m = 2n + 1 \\
\beta^n l_1 \beta^n & m = 2n
\end{cases}
\end{align*}
\]
Lemma 3. Let \( \pi : E \to B \) be a fiber bundle with a section \( s : B \to E, \overline{b} = s(b) \).

Then \( \gamma : \pi_1(E, \overline{b}) \cong \pi_1(F_0, \overline{b}) \times \pi_1(B, b) \) with the semidirect product structure given
by monodromy action \( \pi_1(B, b) \odot \pi_1(F_0, \overline{b}) \). Moreover, \( \gamma^{-1} \gamma \) is identified with \( \text{SO}^2 \).

Corollary 4. When \( E = \{ 0 < x < 2 \} \setminus \{ y = x^m \} \), \( B = \{ 0 < x < 2 \} \), \( b = \overline{b} = N \).

we have \( \pi_1(E, \overline{b}) \cong \langle l_1, l_{-1} \rangle \times \mathbb{Z}[x] \), with monodromy described in
Proposition 5. Here \( \langle l_1, l_{-1} \rangle \) is the free group on generators \( l_1 \) and \( l_{-1} \).

Finally, note \( C^2 \setminus \{ y = x^m \} \) homotopic
\( \{ 0 < x < 2 \} \setminus \{ y = x^m \} \cup \{ y = 0 \} \cup \{ y > 0 \} = E \cup L_0 \),
but with \( L_0 \) adding in, the section \( s : B \to E, \pi \to (x, N) \) extends across origin
therefore by Lemma 3, \( \gamma^{-1} \gamma = 0 \in \pi_1(C^2 \setminus \{ y = x^m \}, \overline{b}) \).

\( j : E \to E \cup L_0 \),
this shows:

Theorem 6. \( \pi_1(C^2 \setminus \{ y = x^m \}, \overline{b}) = \langle l_1, l_{-1} \rangle \) \( \overline{[l_1, l_{-1}] = [l_1, l_{-1}] = [l_1, l_{-1}], [l_1, l_{-1}] = [l_1, l_{-1}] \rangle \)
\( \leq \langle l_1, l_{-1} \rangle \)
\( \underbrace{l_1, l_{-1}, l_1, l_{-1}, \ldots}_{m} = \underbrace{l_1, l_{-1}, l_1, l_{-1}, \ldots}_{m} \).

In case \( m = 3, 4, 6 \), this is exactly the Brinn group \( B_m \) for \( A_2, B_m, C_2 \).
This settles Breenon's theorem in the case where root system has \( r \geq 2 \).

Higher rank case. We need more delicate study on the relation b/w Weyl

Group action on \( \text{fix} \) and \( \text{fix} \). Where \( \text{fix} \) he is identified with the real part.
To simplify our notation, let \( V = h_e \) and \( V = \text{fix} \), so \( V = V \pm iV \).
Let $C_0$ be a Weyl chamber in $V$. We define a stratification

$$C = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_{k-1} = \emptyset$$

where $C_i = \{ x \in \mathbb{C} \mid x \text{ lies in at most } i \text{ walls of } C \}$. Denote $g : V \to V$ the projection to the first factor. Then we define $X_i = (g^{-1}(C_i) - U_{H_0}) / W$, with stratification

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_{k-1} = \overline{\text{reg}} \cap W = h_{\text{reg}}(W)$$

by open subsets of real codimension $i+1$ for $0 \leq i \leq k-1$.

The following lemma will show that

1. $\pi_1(X_1, x) = \pi_1(X_1, x)$

Lemma: Let $M$ be a smooth manifold, $N \subset M$ a closed submanifold of codimension $\geq 3$. Then the inclusion $j : N \hookrightarrow M$ induces $j_* : \pi_1(N, x) \to \pi_1(M, x)$.

Proof: Any element in $\pi_1(M, x)$ is represented by a smooth loop $S : S^1 \to M$. By transversality, $V$ can be deformed to be disjoint from $V$, which shows $j_* x$ is surjective.

To show injectivity, if $j_* [a] = 0 \in \pi_1(M, x)$, for some smooth $S' : S^1 \to M$, then $S' \cdot S$ defines a smooth loop $S^1 \to M$. By transversality of $S'$ and $V$, $S'$ can be deformed into a map $S' : S^1 \to N \setminus V$, bounding $S'$, which shows $j_* [a] = 0 \in \pi_1(M, x)$. \qed

2. Let $W_x$ denote the stabilizer of a point $x \in C$. Under Weyl group action $W_x$ acts freely on the fiber $g^{-1}(C_i) - U_{H_0} = \{ x \} \times (C_i - U_{H_0})$. This shows

$$X_i = U \times \{ x \} \times (C_i - U_{H_0})$$
For $x \in C_0$, $W_x = \mathbb{Z}^3$, so $X_0 = C_0 \times \mathbb{R}^2$. 

For $x \in C \setminus C_0$, $W_x = \mathbb{Z}_2$, so $X = \bigcup_{x \in C \setminus C_0} \left( V_{x'} \setminus H_x \right) \times \mathbb{Z}_2$.

Denote $\{H_j\}_{j \in J}$ the real part of the walls of $C_0$. We have $C \setminus C_0 = \bigcup_{j \in J} (V_{H_j} \setminus H_j)$. So every point on $X \setminus X_0$ has an open neighborhood in $X_1$, disjoint from other walls. On the other hand, $V_{x'} \setminus H_x$ is disjoint union of two open half-spaces, with $\mathbb{Z}_2$ gluing them together. Topologically, this add a handle on each boundary. Altogether, one shows that $X_1 = \bigcup_{j \in J} V_{H_j} \setminus H_j$ with generators $y_j$ corresponding to $j$-th handle.

For $x \in C \setminus C_1$, $W_x$ is a rank 2 Weyl group. They can be of type $A_n$, $B_n$, $C_n$, $D_n$ by classification theorem, i.e., $W_x = S_n \times S_{n-1}, S_n \times S_{n-2}, S_n \times S_{n-3}, S_n \times S_{n-4}, S_n \times S_{n-5}, S_n \times S_{n-6}$. If we denote $X_{\mathbb{R}^2}^{jk}$ as the open subset of $X_0$ obtained as adding $H_{j'}\cap H_{i'}$, part to $X_1$. More precisely, $X_{\mathbb{R}^2}^{jk} = X_1 \cup \bigcup_{x \neq H_{j'} \cap H_{i'}} (V_{x'} \setminus (H_{j'} \cup H_{i'})) / W_x$.

Now, we rechoose the candidate on $C_0 \cap V$. By setting $y_j = H_{j'}$, $x \in H_{j'}$, and $z_{i'}$, $\cdots$ $z_{i'}$ other walls of the Weyl chamber, in arbitrary order. In this coordinate, $X_{\mathbb{R}^2}^{jk}$ is identified with a open subset in $C \setminus C_0$ identical by a finite group acting only on the first two coordinate. So we can homotopy the open subset to $z_{i'} z_{i'} = \cdots = z_{i'} z_{i'}$, which reduces to a 2-diml space with quotient by a free action of rank 2 Weyl group. This space is homotopically equivalent to $V_{H_{j'} \cap H_{i'}} / W_{H_{j'} \cap H_{i'}}$, that is, the standard rank two root space whose regular part quotient by a rank two Weyl group. But by Theorem, we have 0.
(3) \( \pi_1(X^*_{\ast}, *) \cong \langle s_j, s_k | s_j s_k s_j s_k \cdots s_j s_k s_j s_k \cdots \rangle \)

(4) \[ X_\ast = \bigcup_{j,k \in I} X^*_{j,k} \]

where \( I = \) index set of wall of Weyl chamber \( C_{\ast} \).

By Van-Kampen theorem, applied to the open cover as above, we have:

\[ \pi_i(X_\ast, *) \cong \langle s_j \cdots s_n | s_j s_k \cdots s_k s_j \cdots \rangle \]

where \( j,k \in I \).

Finally, by (1), we have shown Breeskum's thm. \( \pi_i(Y_{C_{\ast} \setminus C_{\ast}^0}, *) \cong \mathbb{Z} \)

Remark: There is another way to calculate \( \pi_i(C_{\ast} \setminus Y_{=X^*_{\ast}}*, *) \). The idea is to look at the intersection \( (C_{\ast} \setminus Y_{=X^*_{\ast}}) \cap S^2_\mathbb{C} \) for a small 3-sphere in \( \mathbb{R}^4 \cong \mathbb{C}^2 \).

which is a complement of a knot (or in general a link) in \( S^2_\mathbb{C} \).

\[ \begin{cases} 
\text{e.g. } m=3 & \text{trefoil knot} \\
\text{m=4} & \text{link of two type (2,1) knot exterms} \\
\text{m=6} & \text{link of two trefoil knots} 
\end{cases} \]

Reference:

