Brieskorn's theorem on Fundamental Group.

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Let $\mathfrak{g}$ be a simple Lie algebra $/\mathbb{C}$, $\mathfrak{h} \subseteq \mathfrak{g}$ Cartan subalgebra. $\mathfrak{h}_{\mathbb{C}}$ the complexification and W the Weyl group. with action on $\mathfrak{h}$ (and hence to $\mathfrak{h}_{\mathbb{C}}$) generated by reflections. Remove the hyperplanes $\{H_\alpha\}_{\alpha\in R^+}$ fixed by reflections, and denote it as $\mathfrak{h}_{\mathbb{C}}^{reg}$. Therefore Weyl group act on it freely. we will prove:

<u>Theorem (Brieskorn. 1971)</u> $\pi_1(\mathfrak{h}_{\mathbb{C}}^{reg}/W, *) \cong B_W$

Here B_W is the braid group associated to Weyl group. W. defined as

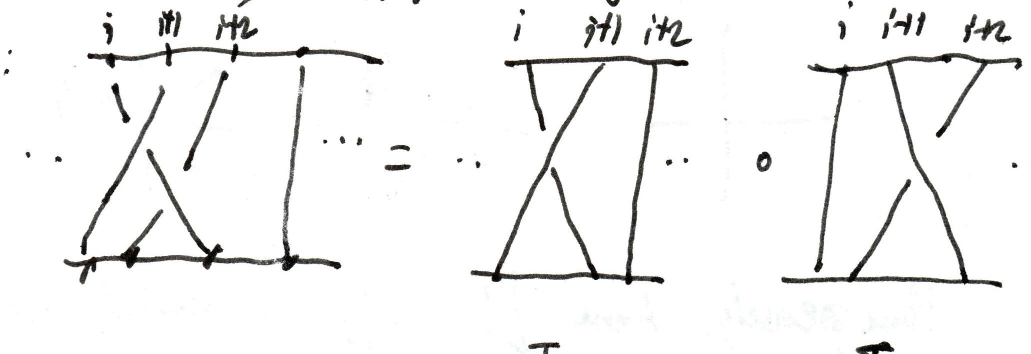
$$B_W = \langle T_i, i\in I \mid \underbrace{T_iT_jT_i\cdots}_{m_{ij}\text{ terms}} = \underbrace{T_jT_iT_j\cdots}_{m_{ij}\text{ terms}} \rangle$$

by fixing a Weyl chamber with I the index set of walls of the chamber.

<u>Example:</u> $\mathfrak{g} = \mathfrak{sl}_n$, $W = S_n$. B_{S_n} is just the Artin Braid group.

$$\langle T_1,\dots,T_{n-1} \mid T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}\ \ {\scriptstyle 1\le i\le n-2},\ \ [T_i,T_j]=e\ \ |i-j|>1 \rangle$$

With strands presentation:



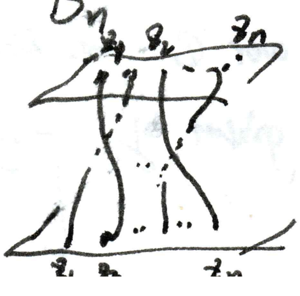
T_i $\qquad$ T_{i+1}

<u>Proof of theorem for $\mathfrak{g}=\mathfrak{sl}_n$</u>: This is known by Fox-Neuwirth in 1962 that Artin's Braid group is isomorphic to the configuration space of unordered distinct n points on $\mathbb{C}$:

$S_n \circlearrowright Y_n \subseteq \mathbb{C}^n$ $\qquad$ $\pi_1(Y_n/S_n, *) \xrightarrow{\cong} B_n$

$\quad \| $

$\mathbb{C}^n \setminus \{(z_1,\dots,z_n) \mid z_i = z_j, i\neq j\}$. $\qquad$ $\gamma \longmapsto$



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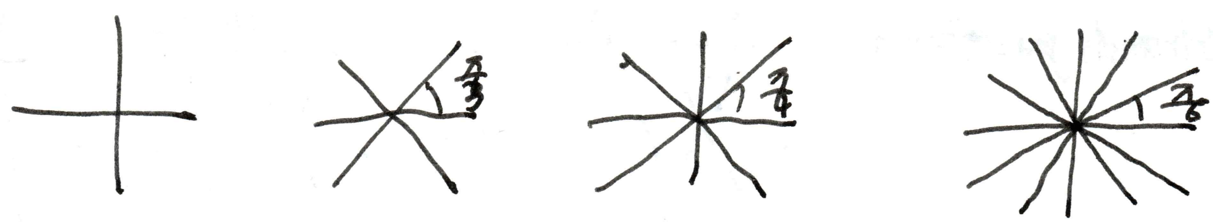
Now, $\mathfrak{h}_{\mathbb{C}}^{reg} = \{(z_1,\cdots,z_n)\in\mathbb{C}^n \mid \sum_{i=1}^n z_i = 0,\ z_j \neq z_k,\ j\neq k\}$ is contained in Y_n as a hyperplane intersection by $\sum_{i=1}^n z_i = 0$, as it intersects $\{z_i = z_j\}$ transversely for each $i\neq j$. We claim that $\mathfrak{h}_{\mathbb{C}}^{reg} \times \mathbb{C} \simeq Y_n$ via $(z_1,\cdots,z_n,t) \mapsto (z_1+t,\cdots,z_n+t)$

Shrinking the line to its origin gives a deformation retract from Y_n to $\mathfrak{h}_{\mathbb{C}}^{reg}$, which is equivariant under S_n-action, which shows $\pi_1(\mathfrak{h}_{\mathbb{C}}^{reg}/S_n, *) \simeq \pi_1(Y_n/S_n, *) \simeq B_n$ $\square$

In general, we will first show the theorem for $rk=2$ root space, and reduce the higher rank case to $rk=2$ case.

<u>$rk=2$</u> There are three such Lie algebras. A_2, B_2, G_2, with Weyl group is $W = \langle s_1, s_2 \mid (s_1s_2)^m = e,\ s_1^2 = e,\ s_2^2 = e\rangle$ $m = 3, 4, 6$. (We can also include $m=2$ for $A_1\times A_1$). Up to rotation of graph, the hyperplanes to be removed in $\mathbb{C}^2$ are



More precisely, choose $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ as standard basis on $\mathbb{C}^2$, the action of W is generated by

$$\underset{\text{rotation}}{\begin{bmatrix}\cos\frac{2\pi}{m} & -\sin\frac{2\pi}{m}\\ \sin\frac{2\pi}{m} & \cos\frac{2\pi}{m}\end{bmatrix}},\quad \underset{\text{reflection}}{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}$$

Working over $\mathbb{C}$, we know we can diagonalize the rotation matrix; so by conjugation W action is given by $\begin{bmatrix}e^{\frac{2\pi i}{m}} & 0\\ 0 & e^{\frac{-2\pi i}{m}}\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

$\mathfrak{h}_{\mathbb{C}}$, denote $\sigma = e^{2\pi i/m}$. $W \curvearrowright \mathbb{C}^2$ by $\sigma: (u,v) \mapsto (\sigma u, \sigma^{-1} v)$

$s: (u,v) \mapsto (v, u)$.

As $\langle\sigma\rangle \subseteq W$ is a normal subgroup. The action by W factors through a cyclic cover $\mathbb{C}^2 \to \mathbb{C}^2/\langle\sigma\rangle$, we will first understand this space. Note that by the cyclic action $\sigma: (u,v) \mapsto (\sigma u, \sigma^{-1} v)$, the invariant polynomials is an ideal generated by (uv, u^m, v^m). By setting $x = uv$, $s = u^m$, $t = v^m$. We have relations $x^m = st$, therefore $\mathbb{C}^2/\langle\sigma\rangle$ is identified with $\{x^m = st\} \subseteq \mathbb{C}^3$.

Now the induced reflection act on it by $(x, s, t) \mapsto (x, t, s)$. By change of variable $s = y - z$, $t = y + z$, we identify $\mathbb{C}^2/\langle\sigma\rangle$ by $\{x^m = y^2 - z^2\}$ with $\mathbb{Z}_2$-action by $z \mapsto -z$ with branching locus $\{x^m = y^2\} \subseteq \mathbb{C}^2$. This shows:

<u>proposition 1</u> $\mathfrak{h}_{\mathbb{C}}^{reg}/W \cong \mathbb{C}^2 \setminus \{y^2 = x^m\}$.

So the problem reduces to calculate fundamental group of the complement of the plane curve $y^2 = x^m$ in $\mathbb{C}^2$

Let $C = \{y^2 = x^m\}$. The projection $\mathbb{C}^2 \setminus C \to \mathbb{C}$, $(x,y) \mapsto x$ has fiber line minus two points everywhere except for at $x = 0$. Denote L_0 the fiber at $x = 0$. Then $\mathbb{C}^2 \setminus (C \cup L_0) \to \mathbb{C}^*$, $(x,y) \mapsto x$ is a fiber bundle with fiber diffeomorphic to $L_1 = \mathbb{C} - \{\pm 1\}$

<u>Monodromy of fundamental group</u>: In general, let $F \hookrightarrow E \xrightarrow{\pi} B$ be a fiber bundle. that is, E and B are smooth manifolds, and for every point $b \in B$, $\exists\, U \subseteq B$ a nbhd of b, such that $E|_U \underset{\text{diffeo.}}{\cong} U \times F$. If we further assume that both F and B are path connected. For $b \in B$, choose $\tilde{b} \in F_b = \pi^{-1}(b)$, then there is an action $\pi_1(B, b) \curvearrowright \pi_1(F_b, \tilde{b})$

Let's assume there is a section $s: B \to \tilde{E}$ (i.e. a smooth map s.t. $\pi \circ s = Id$) and take $\tilde{b} = s(b)$. Let $\gamma: [0,1] \to B$ with $\gamma(0) = \gamma(1)$ be a loop on B. and $\eta: [0,1] \to F_b$, with $\eta(0) = \eta(1) = \tilde{b}$ be a loop on F_b, we want to describe the action of $[\gamma]$ on $[\eta]$.

Pullback $\gamma^* E \to [0,1]$ defines a bundle over $[0,1]$, whose fiber at t is $F_{\gamma(t)}$ with a preferred base point $s(\gamma(t))$, and the data $(F_{\gamma(t)}, s(\gamma(t)))$ varies continuously w.r.t t. Now, we can deform η into a loop η_t on $F_{\gamma(t)}$ based at $s(\gamma(t))$. As $\gamma(0) = \gamma(1) = b$, so $s(\gamma(0)) = s(\gamma(1)) = \tilde{b}$, we have η_1 is a well defined loop in F_b again based at $\tilde{b}$. So we define

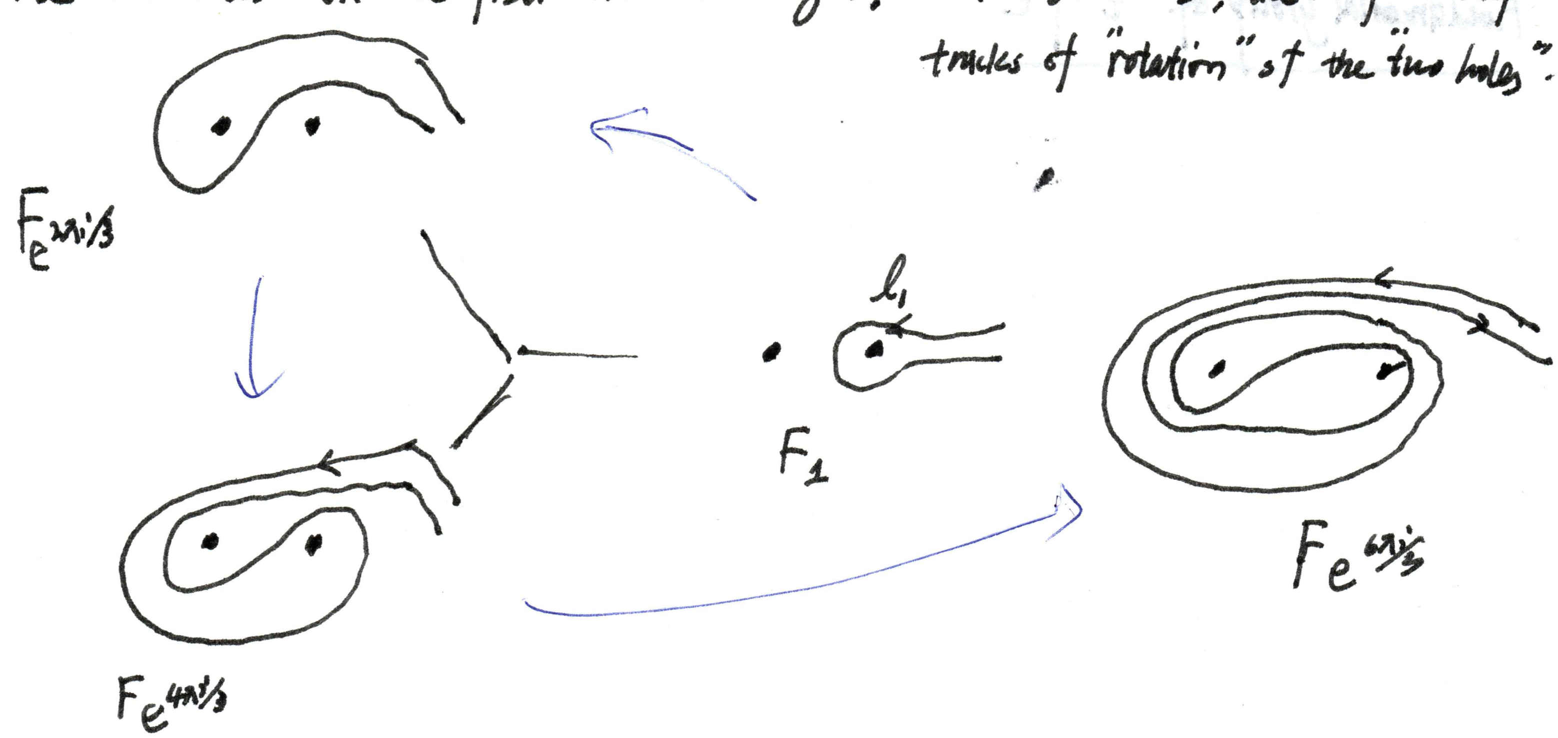
$$\pi_1(B, b) \curvearrowright \pi_1(F_b, \tilde{b})$$

$$([\gamma], [\eta]) \longmapsto [\eta_1] =: [\gamma].[\eta]$$

described as above. By Serre's lifting property, this is well-defined.

Now, go back to our case $\pi: \mathbb{C}^2 \setminus (C \cup L_0) \to \mathbb{C}^*$. By shrinking the base to $B = \{ x \in B | < 2 \}$ and $E := \pi^{-1}(B)$, we obtained our prefered bundle $E \xrightarrow{\pi} B$ with a section $s: B \to E$, $x \mapsto (x, N)$, where N is some large number on the real line such that $N^2 = x^m$ has no solution for $x \in B$. This is our preferred base point on each fiber.

Chose $\gamma: [0,1] \to B$, $t \mapsto e^{2\pi i t}$, to be the loop around origin. We are keeping track of fibers $F_{\exp(2\pi i t)}$ along the unit circle. We found that for each $k = 0, \dots, m$ and on the interval $[\frac{k}{m}, \frac{k+1}{m}]$, as $F_{\exp(\frac{2\pi i k}{m})}$ moves to $F_{\exp(\frac{2\pi i (k+1)}{m})}$ along the interval the "two holes" on the fiber are interchanged. In case $m=3$, the loop l_1 keep tracks of "rotation" of the "two holes".



In fact, this is exactly the monodromy action of γ on l_1. If we define l_{-1} to be the loop as follows

Then it is not hard to see. $[\gamma].[l_1] = [\beta^{-1} l_{-1} \circ \beta]$, where $[\beta] = l_1 \circ l_{-1}$.
In general, we have:

proposition 2. Let $E \xrightarrow{\pi} B$ be as above. and same for γ, l_1, l_{-1}, then

$$[\gamma].[l_1] = \begin{cases} \beta^{-n} l_{-1} \circ \beta^{n} & m = 2n+1 \\ \beta^{-n} l_1 \circ \beta^{n} & m = 2n \end{cases}$$

$$[\gamma].[l_{-1}] = \begin{cases} \beta^{-n-1} \circ l_1 \circ \beta^{n+1} & m = 2n+1 \\ \beta^{-n-1} \circ l_{-1} \circ \beta^{n+1} & m = 2n \end{cases}$$

Lemma 3: Let $\pi: E \to B$ be a fiber bundle with a section $s: B \to E$, $\tilde{b} = s(b)$. then $\varphi: \pi_1(E, \tilde{b}) \cong \pi_1(F_b, \tilde{b}) \rtimes \pi_1(B, b)$ with the semidirect product structure given by monodromy action $\pi_1(B, b) \curvearrowright \pi_1(F_b, \tilde{b})$. Moreover, $\varphi^{-1}(\gamma)$ is identified with $s \circ \gamma$

Corollary 4 when $E = \{0 < |z| < 2\} \setminus \{y^2 = x^m\}$, $B = \{0 < |z| < 2\}$, $b = \tilde{b} = N$. we have $\varphi: \pi_1(E, \tilde{b}) \cong \langle l_1, l_{-1} \rangle \rtimes \mathbb{Z}[\gamma]$, with monodromy described in Proposition 5. Here $\langle l_1, l_{-1} \rangle$ is the free group on generators l_1 and l_{-1}.

Finally, note $\mathbb{C}^2 \setminus \{y^2 = x^m\} \overset{\text{homotopic}}{\simeq} (\{0 < |z| < 2\} \setminus \{y^2 = x^m\}) \cup \{(0, y) \mid y \neq 0\} = E \cup L_0$, but with L_0 adding in, the section $s: B \to E$ $x \mapsto (x, N)$ extends across origin. therefore by Lemma 3, $j_* \varphi^{-1}(\gamma) = 0 \in \pi_1(E \cup L_0, \tilde{b})$ where $j: E \hookrightarrow E \cup L_0$, this shows:

two relations defines the same equation

Theorem 6
$$\pi_1(\mathbb{C}^2 \setminus \{y^2 = x^m\}, *) = \langle l_1, l_{-1} \mid \overbrace{[\gamma].[l_1] = [l_1]}, \overbrace{[\gamma].[l_{-1}] = [l_{-1}]} \rangle$$
$$= \langle l_1, l_{-1} \mid \underbrace{l_{-1} l_1 l_{-1} \cdots}_{m} = \underbrace{l_1 l_{-1} l_1 \cdots}_{m} \rangle$$

In case $m = 3, 4, 6$, this is exactly the Braid group B_W for A_2, B_2, G_2. This settles Brieskorn's theorem in the case where root system has $rk = 2$.

Higher rank case We need more delicate study on the relation b/w Weyl group action on $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{C}}$, where $\mathfrak{h}_{\mathbb{R}} \subseteq \mathfrak{h}_{\mathbb{C}}$ is identified with the real part. To simplify our notation, let $V = \mathfrak{h}_{\mathbb{C}}$, and $V' = \mathfrak{h}_{\mathbb{R}}$, so $V = V' \oplus iV'$

Let C_0 be a Weyl chamber in V'. we define a stratification.

$$C = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_k = \overline{C} \qquad k = \text{rank of rootspace.}$$

where $C_i = \{x \in \overline{C} \mid x \text{ lies in at most } i \text{ walls}\}$. Denote $q: V \to V'$ the projection to the first factor. then we define $X_i := \left(q^{-1}(C_i) - \bigcup_{\alpha \in R_+} H_\alpha\right)/W$. with stratification

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X^{k} = V^{reg}/W \left(= \mathfrak{h}_{\mathbb{C}}^{reg}/W\right) \qquad \text{~~I = index set of walls of } C_0\text{~~}$$

by open subsets of real codimension $i+1$ for $0 \le i \le k-1$.

The following lemma will show that

(1) $\pi_1(\mathfrak{h}_{\mathbb{C}}^{reg}/W, *) = \pi_1(X_2, *)$

Lemma: Let M be a smooth connected manifold, $N \subseteq M$ a closed submanifold of $\text{codim}_{\mathbb{R}} \geq 3$. then the inclusion $j: M \setminus N \hookrightarrow M$ induces $j_*: \pi_1(M \setminus N, *) \cong \pi_1(M, *)$.

Pf: Any element in $\pi_1(M, *)$ is represented by a smooth loop $\gamma: S^1 \to M$. by transversality, γ can be deformed to be disjoint from V, which shows j_* is surjective. To show injectivity. if $j_*[\gamma] = 0 \in \pi_1(M, *)$. for some smooth $\gamma: S^1 \to M \setminus V$. then $\exists$ $g: D \to M$ with $g|_{S^1} = j_*\gamma$. again by transversality of g and V. g can be deformed into a map $D \to M \setminus V$. bounding $[\gamma]$. which shows $[\gamma] = 0 \in \pi_1(M \setminus N, *)$. $\square$

(2) Let W_x denote the stabilizer of a point $x \in \overline{C}$. under Weyl group action. So W_x acts freely on the fiber $q^{-1}(x) - \bigcup_{\alpha \in R_+} H_\alpha = \{x\} \times \left(\underset{\uparrow \text{ imaginary}}{i} V' - \bigcup_{x \in H_\alpha} H_\alpha\right)$. This shows

$$X_j = \bigcup_{x \in C_j} \{x\} \times \left(iV' - \bigcup_{x \in H_\alpha} H_\alpha\right)$$

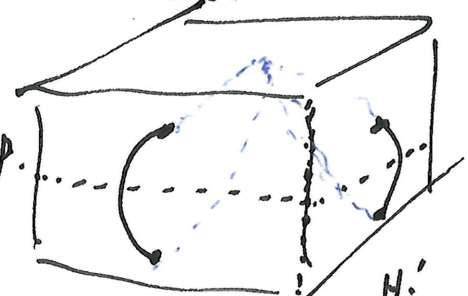
• For $x \in C_0$. $W_x = \{e\}$, so $X_0 = C_0 \times i\mathbb{R}^n \simeq *$

the hyperplane in V whose real part fixes x

• For $x \in C_1 \setminus C_0$. $W_x = \mathbb{Z}_2$. so $X_1 = X_0 \amalg \bigcup_{x \in C_1 \setminus C_0} \{x\} \times (iV' - H_x)/\mathbb{Z}_2$

Denote $\{H_j'\}_{j \in I}$ the real part of the walls of C_0. we have $C_1 \setminus C_0 = \bigcup_j H_j' \setminus (\bigcup_{j,k} H_j' \cap H_k')$

So every point on $X_1 \setminus X_0$ has a open neighborhood in X_1 disjoint from other walls. On the other hand. $iV' - H_x$ is disjoint union of two open half spaces. with $\mathbb{Z}_2$ gluing them together. Topologically. this add a handle on each boundary. Altogether. one shows that $X_1 \simeq \underbrace{S^1 \vee \cdots \vee S^1}_{n}$ with generators g_i corresponding to ith handle.



• For $x \in C_2 \setminus C_1$, W_x is a rank 2 Weyl group. they can be of type $A_1 \times A_1$. A_2. B_2, G_2. by classification theorem. i.e., $W_x = \langle s_1, s_2 \mid s_1^2 = s_2^2 = e, (s_1 s_2)^m = e \rangle$ $m = 2, 3, 4, 6$.

If we denote X_2^{jk} as the open subset of X_2 obtained as adding $H_j' \cap H_k'$ part to X_1. More precisely. $X_2^{jk} := X_1 \amalg \bigcup_{x \in H_j' \cap H_k' \cap C_2} \{x\} \times (iV' - \cancel{H_j \cup H_k} \bigcup_{x \in H_\alpha} H_\alpha)/W_x$

can be 2, 3, 4, 6 hyperplanes

Now. we rechoose the coordinate on $\mathbb{C}^n = V$. by setting $z_1 = H_j$. $z_2 = H_k$. and $z_3, \cdots, z_n$ other walls of the Weyl chamber. in arbitrary order.

In this coordinate, X_2^{jk} is identified with a open subset in $\mathbb{C}^k$ quotient by a finite group acting only on the first two coordinate. So we can homotopy the open subset to $z_3 = z_4 = \cdots = z_n = 0$. which reduces to a 2-dim'l space with quotient by a free action of rank = 2 Weyl group. Such space is homotopic equivalent to V_{reg}^{jk}/W^{jk}, that is. the standard rank two root space whose regular part quotient by a rank two Weyl group. But by Theorem. we have

(3) $\pi_1(X_2^{jk}, *) \cong \langle s_j, s_k \mid \underbrace{s_j s_k s_j \cdots}_{m_{jk} \text{ terms}} = \underbrace{s_k s_j s_k \cdots}_{m_{jk} \text{ terms}} \rangle$

(4) $X_2 = \bigcup_{\substack{j \neq k \\ j,k \in I}} X_2^{jk}$ where I = index set of wall of Weyl chamber C_0.

By Van-Kampen theorem. applied to the open cover as above, we have.

$$\pi_1(X_2, *) \cong \langle s_1, \cdots, s_n \mid \underbrace{s_j s_k s_j \cdots}_{m_{jk} \text{ terms}} = \underbrace{s_k s_j s_k \cdots}_{m_{jk} \text{ terms}}, j \neq k \rangle \simeq B_W$$

Finally, by (1), we have shown Brieskorn's thm. $\pi_1(\mathfrak{h}_{\mathbb{C}}^{reg}/W, *) \simeq B_W$

$\boxtimes$

Remark: There is another way to calculate $\pi_1(\mathbb{C}^2 \setminus \{y^2 = x^m\}; *)$. the idea is to look at the intersect $(\mathbb{C}^2 \setminus \{y^2 = x^m\}) \cap S_\varepsilon^3$ for a small 3-sphere in $\mathbb{R}^4 \cong \mathbb{C}^2$. which is a complement of a knot (or in general a link) in S_ε^3

e.g. $m=3$ trefoil knot

$m=4$ link of two type $(2,1)$ knots on torus

$m=6$ link of two trefoil knots

Reference:

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