

# Stereographic Projection

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April 23, 2022

This semester, we focus on stereographic projection, which is a bijective map from the unit sphere in  $\mathbb{R}^3$  with the north point removed  $S^2 \setminus \{N\}$  to the  $z = 0$  plane. The map is conformal and sends circles on the spheres to circles or the lines on the plane. We explored different proofs of conformality. The first proof verifies that the map preserves the angle formed by two intersecting curves directly by brutal force; The second proof reduces the verification to a planar geometry argument[4]; The third proof verifies an equivalent definition of conformality from the point of view of Riemannian geometry: The pullback Euclidean metric  $|dz|^2$  on the plane differ from the metric on sphere  $g_{S^2}$  by a smooth function, namely,

$$\frac{4|dz|^2}{(1 + |z|^2)^2} = g_{S^2},$$

where  $z = u + iv$  and we identify  $\mathbb{R}^2$  with  $\mathbb{C}$ .

A Möbius transformation is an automorphism of the extended complex plane  $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$  and has the form  $f(z) = \frac{az+b}{cz+d}$  with  $ad - bc \neq 0$ . The stereographic projection extends the north pole and is a homeomorphism  $S^2 \cong \hat{\mathbb{C}}$ . By considering a rotation of the sphere conjugate by stereographic projection, it induces a Möbius transformation on  $\hat{\mathbb{C}}$ . We explored this relationship following the article [1] and the note [3].

As preparation, we reviewed some basic linear algebra and explored basic group theory through examples of 2-by-2 matrix groups, e.g.,  $O(2)$ . We also solved some exercises in [5, Chapter 1] and [2, Section 1.2].

## References

- [1] Douglas N. Arnold and Jonathan Rogness. Möbius transformations revealed. *Notices Amer. Math. Soc.*, 55(10):1226–1231, 2008.
- [2] Theodore W. Gamelin. *Complex analysis*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 2001.
- [3] Online notes. Möbius transformations. [http://www.warwickmaths.com/wp-content/uploads/2020/07/80\\_-Mbius-Transformations.pdf](http://www.warwickmaths.com/wp-content/uploads/2020/07/80_-Mbius-Transformations.pdf), 2021.

- [4] Online notes. Stereographic projection is conformal. <https://people.reed.edu/~jerry/311/stereo.pdf>, 2021.
- [5] Miles Reid and Balázs Szendrői. *Geometry and topology*. Cambridge University Press, Cambridge, 2005.