DEGENERATION OF DUAL VARIETIES

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1. Introduction

The motivation of this article is the following, consider a family of plane conics $C_t = \{xy + tz^2 = 0\} \subset \mathbb{P}^2$ with $t$ varying in a neighborhood of 0. Then for $t \neq 0$, the dual curve is a smooth conic, but when $t = 0$, the conic consists of two lines, whose dual is the set of two points. The dual map on each individual fiber does not need to have the same dimension. However, if we look at the dual curves $C_t$ in family, which is $\{4tuw + w^2\} \subset (\mathbb{P}^2)^*$ and is a flat family. Take limit as $t \to 0$, we still get a degree two curve $\{w^2 = 0\}$. We’d like to use the second situation to define dual curve in the limit.

More generally, let $X_d$ be a degree $d \geq 2$ smooth hypersurface of $\mathbb{P}^{n+1}$ defined by $F_d: X_{d_1}, X_{d_2}$ be smooth hypersurfaces of degree $d_1$ and $d_2$ respectively defined by $F_{d_1}$ and $F_{d_2}$, with $d = d_1 + d_2$.

We require $F_d, F_{d_1}, F_{d_2}$ to be general, so that their common zero locus is a complete intersection. Besides, by Bertini’s theorem, $F^*: = sF_d + F_{d_1}, F_{d_2}$ is smooth for $s \neq 0$ when $|s|$ is small enough. So for such $s \neq 0$, there is a dual map on smooth hypersurface $X^*: = \{F^* = 0\}$

$$D_s : X^* \mapsto (\mathbb{P}^{n+1})^*$$

$$x \mapsto \left(\frac{\partial F^*}{\partial x_0}(x), ..., \frac{\partial F^*}{\partial x_{n+1}}(x)\right)$$

with $\frac{\partial F^*}{\partial x_j}(x) = s \frac{\partial F_d}{\partial x_j}(x) + F_{d_2} \frac{\partial F_{d_1}}{\partial x_j}(x) + F_{d_1} \frac{\partial F_{d_2}}{\partial x_j}(x)$, $j = 0, ..., n + 1$ by direct computation.

The image $(X^*)^*$ is called the dual variety of $X^*$ and it is well know that it is a hypersurface of degree $m = d(d-1)^n$ in the dual space. So this defines a rational section $\mu$ on the sheaf $S^m(V^*) \otimes \mathcal{O}_{\Delta}$ over $\Delta$ which has possibly a pole along $s = 0$ where $V = \mathbb{C}^{n+2}$, but by multiplying by a suitable power of $s$, we can assume the section $\mu$ is regular and $\mu(0) \neq 0$. This will not change the defining hypersurface in projective space, so it defines a hypersurface.

Definition. Define $(X^0)^*$ to be the projective hypersurface $\{\mu = 0\} \subseteq (\mathbb{P}^{n+1})^*$ and call it the dual variety in the limit associated to the family $sF_d + F_{d_1}, F_{d_2}$.

Our purpose of this section is to understand different connected components of $(X^0)^*$ and its corresponding multiplicities.

Date: July, 27, 2020.
Theorem 1. The dual variety in the limit $(X^0)^*$ associated to the family $\{F^s = 0\}_{s \in \Delta}$ is a reducible hypersurface in $(\mathbb{P}^{n+1})^*$. When $n = 1$, $(X^0)^*$ is a curve with components:

I. dual variety of $X_{d_1}$ and $X_{d_2}$, reduced;
II. dual variety of $X_{d_1} \cap X_{d_2}$, each component acquiring with multiplicity two.

When $n \geq 2$, $(X^0)^*$ consists of components of type I, II as above, together with

III. dual variety of $X_{d_1} \cup X_{d_2}$, each component acquiring with multiplicity two.

Moreover if $d_i = 1$, then the corresponding component $X_{d_i}^*$ is trivial (dual variety of a hyperplane is a point) and the components in (II) and (III) are cones over dual variety in the hyperplane $X_{d_i}$.

Example 1. Consider the family of smooth cubic curves degenerate into a conic $Q$ union a line $L$ intersecting transversely, for an explicit example

$$F^s(x_0, x_1, x_2) = s(x_0^3 + x_1^3 + x_2^3) + x_0(x_0^2 + x_1^2 + x_2^2) = 0.$$ 

Let $(u, v, w)$ be the coordinates on dual space $(\mathbb{P}^2)^*$, then the dual variety in the limit $(X^0)^*$ consists of (1) a conic as dual curve of $Q$; (2) lines $v = \pm w$ with multiplicity two.

Since the dual curve of a smooth cubic has degree 6, the decomposition reads $6 = 2 + 2 \times 2$.

Example 2. Consider the family $sF_5 + F_3F_2 = 0$ of quintic surface degenerate to quadric surface union a cubic surface.

The dual variety in the limit $(X^0)^*$ consists of (1) dual variety of cubic, which has degree 12; (2) dual variety of quadric, which is again a quadric; (3) dual variety of $X_3 \cap X_2$, which has degree 18, with multiplicity two; (4) dual variety of $X_5 \cap X_3 \cap X_2$, which is the set of hyperplanes through the 30 points, namely union of 30 hyperplanes in $(\mathbb{P}^3)^*$.

Since the dual variety of quintic surface has degree 80, the decomposition reads $80 = 12 + 2 + 2 \times 18 + 30$.

2. Multiplicity Counting

Assume $M, N$ are complex varieties and are flat over $\Delta$ via $f, g$. $h$ is a regular map making the diagram commutes:

$$\begin{array}{ccc}
M & \xrightarrow{h} & N \\
\downarrow{f} & & \downarrow{g} \\
\Delta & \downarrow{g} & \\
\end{array}$$

Let $Z$ be a component of $M_0 = f^{-1}(0)$, let’s assume that $h(Z)$ is a component of $N_0 = g^{-1}(0)$.

The multiplicity $m_Z$ of $Z$ is the order of vanishing $f^*t$ on $Z$, where $t$ is the local equation of $0 \in \Delta$. Similarly the multiplicity $m_{h(Z)}$ of $h(Z)$ is the order of vanishing of $g^*t$ on $h(Z)$. Then by $f^* = h^* \circ g^*$, we have the equality

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\[ n_Z = k \cdot m_{h(Z)}, \]

where \( k \) is the ramification index of \( h \) at the component \( Z \), which can be defined as following:

Choose \( p \in Z \) a general point, and \( \Delta_p \) a holomorphic disk in \( M \) which intersects \( Z \) transversly at \( p \), then the restriction \( h|_{\Delta_p} \) is a \( k \)-to-1 map onto its image.

So we immediately have the following argument:

**Proposition 1.** If \( h \) has ramification index one along \( Z \), then then \( n_{h(Z)} \) coincides with \( m_Z \).

### 3. Proof of Theorem 1

We define the total space of family

\[ (3) \quad \mathcal{X} = \{sF_d + F_{d_1}F_{d_2} = 0\} \subset \Delta \times \mathbb{P}^{n+1} \]

over the disk \( \Delta \) with \( \pi : \mathcal{X} \to \Delta \) the projection map. So the fiber over \( s \) is the hypersurface \( \{F^s = 0\} \) and the special fiber \( \{F^0 = F_{d_1}F_{d_2}\} \) is reducible. The total space \( \mathcal{X} \) is singular along \( S := \{s = 0, F_d = 0, F_{d_1} = 0, F_{d_2} = 0\} \) since it has local analytic equation \( sx + yz = 0 \). There is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\mathcal{D}} & \Delta \times (\mathbb{P}^{n+1})^* \\
\downarrow \pi & & \downarrow \pi_1 \\
\Delta & & \Delta
\end{array}
\]

where \( \mathcal{D} : (s, p) \mapsto (\frac{\partial F^s}{\partial x_0}(p), \ldots, \frac{\partial F^s}{\partial x_{n+1}}(p)) \) is the dual map on each fiber, which is regular outside the locus \( C := \{s = 0, F_{d_1} = 0, F_{d_2} = 0\} \).

Identify \( S \) (resp. \( C \)) with the complete intersection \( X_d \cap X_{d_1} \cap X_{d_2} \) (resp. \( X_{d_1} \cap X_{d_2} \)) in \( \mathbb{P}^{n+1} \). We want to reach a diagram as we discussed in the previous section. To do this, we need to resolve the singular locus of \( \mathcal{X} \) and the indeterminacy locus of \( \mathcal{D} \). So we blowup \( \mathcal{X} \) along \( S \) and then blowup the strict transform of the indeterminacy locus \( C \) to get a smooth total space \( \tilde{\mathcal{X}} \) with regular dual map \( \tilde{\mathcal{D}} \), and reach a diagram

\[
\begin{array}{ccc}
\tilde{\mathcal{X}} & \xrightarrow{\tilde{\mathcal{D}}} & \Delta \times (\mathbb{P}^{n+1})^* \\
\downarrow \lambda & & \downarrow \pi_1 \\
\mathcal{X}' & \xrightarrow{\mathcal{D}'} & \Delta \times (\mathbb{P}^{n+1})^* \\
\downarrow \pi' & & \downarrow \pi_1 \\
\Delta & & \Delta
\end{array}
\]

where \( \pi' \) is the composite of \( \pi \) and the blowup \( \sigma : \mathcal{X}' \to \mathcal{X} \). We first prove that
Proposition 2. $\tilde{D}$ has multiplicity index one on each component $\tilde{\mathcal{X}}^0$.

Proof. This is due to both $\pi$ and $\pi_1$ are projections to the first factors $\Delta$, moreover $D$ is identity map on this factor. □

Proposition 3. The special fiber $\tilde{\mathcal{X}}^0 := (\lambda \circ \pi')^{-1}(0)$ has the following irreducible components:

I. strict transforms of $X_{d_1}$ and $X_{d_2}$, reduced;

II. the exceptional divisor $\tilde{C}$ over the strict transform of $C$, multiplicity two;

III. the exceptional divisor $\tilde{S}$ over $S$, reduced.

Moreover, their image under $\tilde{D}$ are corresponding dual varieties of type I-III stated in Theorem 1.

Proof. It suffices to prove of type II and III. According to Proposition 2, it suffices to show $\tilde{C}$ and $\tilde{S}$ have the corresponding multiplicities, and their image are the corresponding dual varieties of type II and III.

The local analytic equation of $q_0 \in C$ in $\tilde{\mathcal{X}}$ is,

$$u = 0, \quad v = 0$$

in the hypersurface $\{sf + uv = 0\} \subseteq \Delta^3_{s,u,v} \times \Delta^{n-1}$, where $f = f(s,u,v)$ is an analytic function with $f(0,0,0) \neq 0$ and the last of $n-1$ variables are free. If $q_0 \in C \backslash S$, then it has a neighborhood unaffected by the first blowup $\sigma$, so the multiplicity of $\tilde{C}$ is the same as the multiplicity of the exceptional divisor of blowup of $(0,0,0)$ of $\{sf + uv = 0\} \subseteq \Delta^3_{s,u,v}$, which is two by straightforward computation.

Now let’s prove the image of $\tilde{C}$ under $\tilde{D}$ is the dual variety of $X_{d_1} \cap X_{d_2}$. The total transform of $\lambda$ has local analytic equation over a point $q_0 \in C \backslash S$

$$sf + uv = 0, \quad \text{rank} \begin{bmatrix} \alpha & \beta & \gamma \\ u & v & s \end{bmatrix} \leq 1, \quad (\alpha, \beta, \gamma) \in \mathbb{P}^2.$$

Set $\beta = 1$, so $\alpha v = u$, $s = v \gamma$, $v(\gamma f + \alpha v) = 0$ and the map is

$$\tilde{D} : (\gamma, v, \alpha) \mapsto \left( \frac{\partial F_{d_1}}{\partial x_j}(q) + \frac{\partial F_{d_2}}{\partial x_j}(q) + \alpha \frac{\partial F_{d_2}}{\partial x_j}(q) \right)_{j=0,...,n+1} \in (\mathbb{P}^{n+1})^*,$$

where the local coordinate of $q$ depends on $\gamma, v, \alpha$. As $v$ goes to zero, by equation $\gamma f + \alpha v = 0$, $\gamma$ goes to 0 and we have the dual map on the exceptional divisor

$$\tilde{D}(q_0) = \left( \frac{\partial F_{d_1}}{\partial x_j}(q_0) + \alpha \frac{\partial F_{d_2}}{\partial x_j}(q_0) \right)_{j=0,...,n+1} \in (\mathbb{P}^{n+1})^*.$$

Similarly on chart $\alpha = 1$. It follows that the image at $q_0 \in X_{d_1} \cap X_{d_2} \backslash X \cap X_{d_1} \cap X_{d_2}$ is the set of linear combination of normal vectors along $F_{d_1}$ and $F_{d_2}$ at $q_0$, which form a Zariski open dense subset of $(X_{d_1} \cap X_{d_2})^*$. This finish the proof of the case II.
In case III (only exist when \( n \geq 2 \)), we choose a point \( p_0 \in S \), then \( S \) has local analytic equation in \( \mathcal{X} \)

\[
x = 0, \ y = 0, \ z = 0
\]
in \( \{sx + yz = 0\} \subseteq \Delta^4_{s,x,y,z} \times \Delta^{n-2} \) where the last \( n - 2 \) variables are free. Since the exceptional divisor \( \tilde{S} = \sigma^{-1}(S) \) is generically unaffected by the second blowup \( \lambda \), the multiplicity of \( \tilde{S} \) is computed over a general point \( p_0 \in S \). Again, this multiplicity coincides with the multiplicity of the exceptional divisor of the blowup of origin of \( sx + yz = 0 \), which is one.

Finally, let’s show that the image of \( \tilde{S} \) under \( \mathcal{D} \) is the dual variety of \( X_d \cap X_{d_1} \cap X_{d_2} \). (By abuse of notation, \( \tilde{S} \) is also denoted as its strict transform under the blowup \( \lambda \).) The total transform of \( \sigma \) has local equation over a general point \( p_0 \in S \)

\[
sx + yz = 0, \ \mathrm{rank} \begin{bmatrix} \alpha & \beta & \gamma & \delta \\ s & x & y & z \end{bmatrix} \leq 1, \ (\alpha, \beta, \gamma, \delta) \in \mathbb{P}^3. \tag{5}
\]

If we choose an affine chart \( \alpha = 1 \), then the equation \( (5) \) becomes \( x = s\beta, y = s\gamma, z = s\delta \) and \( s^2(\beta + \gamma\delta) = 0 \), so by substitution and scaling in projective coordinate, the dual map \( (1) \) becomes

\[
\mathcal{D}': (s; \gamma, \delta) \mapsto \left( \frac{\partial F_d}{\partial x_j}(p) + \gamma \frac{\partial F_{d_1}}{\partial x_j}(p) + \delta \frac{\partial F_{d_2}}{\partial x_j}(p) \right)_{j=0,...,n+1} \in (\mathbb{P}^{n+1})^* \tag{6}
\]

around a point \( p_0 \in S \) and the local coordinate of \( p \) depends on \( s, \gamma, \delta \). So \( (6) \) implies that the dual map extends to (an Zariski open subset of) \( \sigma^{-1}(S) \) by

\[
S \times \mathbb{C}^2 \ni (p_0, \gamma, \delta) \mapsto \left( \frac{\partial F_d}{\partial x_j}(p_0) + \gamma \frac{\partial F_{d_1}}{\partial x_j}(p_0) + \delta \frac{\partial F_{d_2}}{\partial x_j}(p_0) \right)_{j=0,...,n+1} \in (\mathbb{P}^{n+1})^*. \tag{7}
\]

Note the map is well-defined on this chart due to the assumption that \( S \) is smooth complete intersection, so three normal directions \( \frac{\partial F_d}{\partial x_j}, \frac{\partial F_{d_1}}{\partial x_j}, \frac{\partial F_{d_2}}{\partial x_j} \) are linearly independent. This shows that the image of \( \tilde{S} \) under \( \mathcal{D} \) contains a Zariski dense subset of \( (X_d \cap X_{d_1} \cap X_{d_2})^* \), so it has to be the whole dual variety.

\[
\square
\]

Since the dual varieties \( \{(X^*)_s\}_{s \in \Delta} \) as we defined in the previous section is a flat family, so in particular each member has the same degree. In what follows, we will show that the sum of degree of the components of three types agrees with degree of the nearby fiber. This proves that \( (X^0)_* \) has no other components, therefore will complete the proof of Theorem 1.

**Proposition 4.** The following identities holds:

\[
\deg(X^*_d) = \deg(X^*_{d_1}) + \deg(X^*_{d_2}) + 2 \deg((X_{d_1} \cap X_{d_2})^*), \text{ if } n = 1; \tag{8}
\]

\[
\deg(X^*_d) = \deg(X^*_{d_1}) + \deg(X^*_{d_2}) + \deg((X_d \cap X_{d_1} \cap X_{d_2})^*), \text{ if } n \geq 2.
\]
Proof. First of all $X^*_d$ has degree $d(d-1)^n$, and $X^*_{di}$ has degree $d_id_i(i-1)^n$, $i = 1,2$. So $n = 1$ case is the consequence of the identity

$$d(d-1) = d_1(d_1-1) + d_2(d_2-1) + 2d_1d_2,$$

which one can readily check by hand.

For $n \geq 2$ case, we need a formula for dual variety of a complete intersection. According to page 362 of Kleiman’s *The enumerative theory of singularities* [1], If $Y \subset \mathbb{P}^N$ is a smooth variety of dimension $m$ with dual $Y^*$ being a hypersurface, then $\deg(Y^*)$ coincides with $\int_Y s_m(E)$, with $E = N_{Y|\mathbb{P}^N}^* \otimes \mathcal{O}_Y(1)$ where $s_m$ is the Segre class and $N_{Y|\mathbb{P}^N}^*$ is the conormal bundle. This uses the fact that the dual variety $Y^*$ is the image of $\mathbb{P}_Y(E)$ and $\deg(Y^*)$ is the $(m-1)$-fold intersection of hyperplane class with $Y^*$. So we have

**Lemma 1.** Let $Y$ be a complete intersection of type $(d_1,...,d_k)$ of dimension $n$, so $E = \oplus_i \mathcal{O}_Y(1-d_i)$ and $\deg(Y^*)$ coincides with the coefficient of $h^n$ of the polynomial

$$\prod_{i=1}^k (1 - (d_i - 1)h)^{-1} \prod_{i=1}^k d_i.$$

Apply the formula to $X_d \cap X_{d_1} \cap X_{d_2}$, which has dimension $n-3$, so we get its degree of dual variety

$$\deg((X_d \cap X_{d_1} \cap X_{d_2})^*) := N_{d,d_1,d_2}^{n-3} = \sum_{i+j+k=n-3 \atop i,j,k \geq 0} (d_1 - 1)^i(d_2 - 1)^j(d_1-1)^k d_1d_2d.$$

Similarly to the complete intersection $X_{d_1} \cap X_{d_2}$ of dimension $n-2$, the degree of its dual variety is

$$\deg((X_{d_1} \cap X_{d_2})^*) := N_{d_1,d_2}^{n-2} = \sum_{i+j=n-2 \atop i,j \geq 0} (d_1 - 1)^i(d_2 - 1)^jd_1d_2.$$

So the proof of Corollary 4 reduces to show the equality

$$d(d-1)^n = d_1(d_1-1)^n + d_2(d_2-1)^n + N_{d,d_1,d_2}^{n-2} + 2N_{d_1,d_2}^{n-1}$$

We prove by induction on $n$. The base case is $n = 2$, one readily check the following identity holds

$$d(d-1)^2 = d_1(d_1-1)^2 + d_2(d_2-1)^2 + 2(d_1 + d_2 - 2)d_1d_2 + dd_1d_2.$$

We want to show the equality (9). By assuming the equality is true for $n-1$ case, so we have

$$d(d-1)^n = d(d-1)^{n-1}(d-1) = [d_1(d_1-1)^{n-1} + d_2(d_2-1)^{n-1} + N_{d,d_1,d_2}^{n-3} + 2N_{d_1,d_2}^{n-2}](d-1)$$

$$= d_1(d_1-1)^n + d_2(d_2-1)^n + d_1d_2[(d_1-1)^{n-1} + (d_2-1)^{n-1}] + (d-1)(N_{d,d_1,d_2}^{n-3} + 2N_{d_1,d_2}^{n-2}).$$
So it reduces to show the equality

\[ N_{d,d_1,d_2}^{n-2} + 2N_{d_1,d_2}^{n-1} = (d-1)(N_{d,d_1,d_2}^{n-3} + 2N_{d_1,d_2}^{n-2}) + d_1d_2[(d_1-1)^{n-1} + (d_2-1)^{n-1}], \]

which is a consequence of identity \((d-1)N_{d,d_1,d_2}^{n-3} = N_{d,d_1,d_2}^{n-2} - dN_{d_1,d_2}^{n-2}\) with direct computation. This finishes the proof. □

REFERENCES