O-MINIMAL CHOW'S THEOREM

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1. Introduction

The classical Chow's theorem states that closed analytic subvarieties of projective space $\mathbb{P}^n$ are algebraic. However, the statement is in general false when the ambient space is replaced by $\mathbb{C}^n$. For example, the graph of the entire function $z \mapsto e^z$ is not zero locus of any polynomial. A key feature is that this function does not behave well at infinity. On the other hand, an o-minimal structure specifies a class of "tame" subsets of $\mathbb{R}^{2n}$ (identified with $\mathbb{C}^n$) with strong finiteness properties which in particular excludes functions like e^z. This leads to a theorem due to Peterzil and Starchenko: Let X be a closed analytic subvariety $\mathbb{C}^n$ and the underlying set of X is definable in an o-minimal structure, then X is algebraic.

2. O-Minimal Structure

Definition. A *structure* on the real field $(\mathbb{R}, +, \cdot)$ is a collection $\mathcal{S} = (S_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$:

(1). S_n is a boolean algebra of subsets of $\mathbb{R}^n$;

(2). S_n contains the diagonal $\{(x_1, ..., x_n) \in \mathbb{R}^n | x_i = x_j\}$;

(3). If $A \in S_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A$ belongs to S_{n+1};

(4). If $A \in S_{n+1}$, then the projection of A on the first n-coordinates is in S_n;

(5). S_3 contains the graphs of addition and multiplication.

Moreover, the structure $\mathcal{S}$ is called *o-minimal* if

(6). S_1 only consists of semialgebraic sets of $\mathbb{R}$, that is, finite unions of points and intervals.

Here a boolean algebra of sets on S_n is equivalent to S_n is closed under finite union, intersection and complement. Moreover, the set of all structures on $\mathbb{R}$ has a natural partial order given by inclusion. If $\mathcal{S}$ and $\mathcal{S}'$ are two structure, then $\mathcal{S} \subseteq \mathcal{S}' \Leftrightarrow S_n \subseteq S'_n$ for all n.

We set S_n to be the set of *algebraic sets* in $\mathbb{R}^n$ consisting of $\{x \in \mathbb{R}^n | f(x) = 0\}$ where $f \in \mathbb{R}[x_1, .., x_n]$. One can readily check that condition (2), (3) and (5) hold for $(S_n)_{n\in\mathbb{N}}$, but condition (1) and (4) fail: complement of $\{0\}$ in $\mathbb{R}$ is not zero locus of any polynomial; algebraic sets are not closed under projection, e.g., the projection of $\{(x, y) \in \mathbb{R}^2 | xy = 1\}$ to x-axis is $\mathbb{R}\backslash\{0\}$. This phenomenon comes from that restriction of linear projection being

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non-proper, which we will address later in the proof of our main theorem. So algebraic sets do not form a structure.

However, we can consider the structure generated by algebraic sets, that is, we include all complements and projections of algebraic sets, and we also allow their finite union, intersection, etc., then we obtain a (minimal) structure which contains all algebraic sets, where a typical element is a finite union of

$$\{x \in \mathbb{R}^n | f(x) = 0, g_1(x) > 0, ..., g_k(x) > 0\},$$

with $f, g_1, ..., g_k \in \mathbb{R}[x_1, ..., x_n]$ and it is called a *semialgebraic set*. This is due to Tarski-Seidenberg theorem, which states that projection of semialgebraic sets are still semialgebraic. Moreover, condition (6) comes for free and it follows that

Example 1. Semialgebraic sets form an o-minimal structure, denoted as $\mathbb{R}_{\text{sa}}$.

Actually, as a result of the definition, every o-minimal structure has to contain semialgebraic sets. Namely, $\mathbb{R}_{\text{sa}}$ is the smallest o-minimal structure. There are examples of structure that are not o-minimal, for example, all subsets of $\mathbb{R}^n$, but in general they lose tameness properties that we will mention at the end of this section. In the rest of notes, all the structure will be o-minimal unless otherwise stated.

Definition. We call a set $A \subset \mathbb{R}^n$ *$\mathcal{S}$-definable* if $A \in S_n$. Moreover, if $A \in S_n$, a function $f : A \to \mathbb{R}^m$ is called *$\mathcal{S}$-definable* if its graph

$$\Gamma(f) = \{(x, f(x)) \in \mathbb{R}^{n+m} | x \in A\}$$

is $\mathcal{S}$-definable.

For example, $x \mapsto \sqrt{1 - x^2}$ for $|x| \leqslant 1$ is $\mathbb{R}_{\text{an}}$-definable, but $x \mapsto e^x$ is not (in any interval).

Let $\mathcal{F}$ be a set of real valued functions or subsets of $\mathbb{R}^n$, one can define the structure $\mathbb{R}_{\mathcal{F}}$ to be the structure generated by $\mathcal{F}$. As an example that has been discussed, taking $\mathcal{F}$ to be set of all real coefficients polynomials will produce $\mathbb{R}_{\text{sa}}$. Accordingly, it produces various kinds of (o-minimal) structures.

Example 2. $\mathbb{R}_{\text{exp}}$ is an o-minimal structure (due to Wilkie), where $\mathcal{F}_{\text{exp}}$ consists of the graph $\{(x, e^x | x \in \mathbb{R})\}$ of the exponential function together with all polynomials.

Note that by reflection of the graph of e^x, one see that $x \mapsto \log(x), x > 0$ is $\mathbb{R}_{\text{exp}}$-definable. For any $r > 0$ (especially irrational r), $x \mapsto x^r, x > 0$ is $\mathbb{R}_{\text{exp}}$-definable For $r > 0$, since $x^r = \exp(r \log(x))$.

Example 3. $\mathbb{R}_{\text{an}}$ is an o-minimal structure (due to van den Dries), where

$$\mathcal{F}_{\text{an}} = \{f | f = g|_{[-1,1]^n}, g \text{ is a real analytic function defined on an open neighborhood of } [-1,1]^n\}.$$

A typical element in $\mathbb{R}_{\text{an}}$ is a subanalytic sets in the real projective space $\mathbb{R}P^n$, where we identify $\mathbb{R}^n$ with an open subset of $\mathbb{R}P^n$ via

$$(x_1, ..., x_n) \mapsto [1, x_1, ..., x_n].$$

The function $x \mapsto \arctan(x)$, $x \in \mathbb{R}$ is $\mathbb{R}_{\text{an}}$-definable, but $x \mapsto e^x, x \in \mathbb{R}$ is not.

Example 4. $\mathbb{R}_{\text{an, exp}}$ is an o-minimal structure (due to Miller and van den Dries), where $\mathcal{F}_{\text{an, exp}} = \mathcal{F}_{\text{an}} \cup \mathcal{F}_{\text{exp}}$. E.g., the complex exponential function $z \mapsto e^z$ restricted to $(-\infty, \infty) \times [-1, 1]$ is $\mathbb{R}_{\text{an,exp}}$-definable.

There are many recent application of this o-minimal structure in algebraic geometry, for example, it is used in Bakker, Klingler and Tsimerman's proof on Cattani-Deligne-Caplan's theorem on algebraicity of Hodge locus [3]; It is also used in Bakker, Brunebarbe and Tsimerman's proof of the Griffiths conjecture [2]. The sauce is, period maps of variation of Hodge structures are definable in $\mathbb{R}_{\text{an,exp}}$.

2.1. **Properties of O-minimal Structures.** In this subsection, we discuss some basic properties of o-minimal structures which will be used in the next section. Note that only theorem 1 will really need o-minimality assumption and the rest propositions hold for a structure in general. One can refer to [11] for detailed discussion.

Proposition 1. *Let $\mathcal{S}$ be a structure, then*

(1). *The image and preimage of a definable set under a definable map is definable;*

(2). *The sum, product and composition of two definable functions are definable.*

Proof. Both statements are directly followed from the definition. For (1), if $f : A \to \mathbb{R}^m$ is a definable function with domain $A \subset \mathbb{R}^n$, then its image is the projection of the definable set $\Gamma(f)$ to the second factor, therefore $f(A)$ is definable. Let $B \subseteq \mathbb{R}^m$ definable, then $f^{-1}(B)$ is the projection to the first factor of the intersection of $\Gamma(f) \cap \mathbb{R}^n \times B$.
For (2), the argument of sum and product follows from property (5) of the structure, so let's prove the argument for composite. Let $f : A \to \mathbb{R}^m$, and $g : B \to \mathbb{R}^p$ be two definable functions with $f(A) \subseteq B \subseteq \mathbb{R}^m$, then the graph $\Gamma(g \circ f)$ is the projection to $A \times \mathbb{R}^p$ of

$$W = \{(x, y_1, y_2, z) \in A \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^p | f(x) = y_1,\ g(y_2) = z,\ z_1 = z_2\}.$$

W is the intersection of $\Gamma(f) \times \Gamma(g)$ with hyperplane $y_1 = y_2$. □

As an application, the functions $x \mapsto \sin(x), x \in \mathbb{R}$ and complex exponential function $z \mapsto e^z, z \in \mathbb{C}$ are not definable in any o-minimal structure, since the preimages of 1 under both functions are infinte and discrete, which violate the o-minimal criterion.

Proposition 2. *Let $A \subseteq \mathbb{R}^n$ be a definable set with respect to a structure $\mathcal{S}$ (with $<$ being definable), then with respect to the euclidean topology, its closure $\bar{A}$, interior A°, and frontier* $\text{fr}(A) := \bar{A} \backslash A$ *are all $\mathcal{S}$-definable.*

Proof. If suffices to prove for the closure. By definition

$$\bar{A} = \{x \in \mathbb{R}^n | \forall \varepsilon \in \mathbb{R}, \exists y \in A \subseteq R^n, \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2\}.$$

As we are familiar with writing $\exists$ as projection, but we don't know how to write $\forall$ set theoretically. The good news is that there is a logic formula $\forall \varepsilon, P(\varepsilon) \Leftrightarrow \neg \exists \varepsilon \neg P(\varepsilon)$, where $P(\varepsilon)$ is a statement on ε. If we set

$$B = \{(x, \varepsilon, y) \in \mathbb{R}^n \times \mathbb{R} \times A | \sum_{i=1}^{n} (x_i - y_i)^2 < \varepsilon^2\},$$

and let $\pi_{1,2}$ denote the projection to the $\mathbb{R}^n \times \mathbb{R}$ and π_1 to $\mathbb{R}^n$, it follows that

$$\bar{A} = \mathbb{R}^n - \pi_1(\mathbb{R}^n \times \mathbb{R} - \pi_{1,2}(B))$$

is definable. □

Theorem 1. *If we fix an o-minimal structure $\mathcal{S}$, there are several tameness properties of definable sets* [11]*:*
(i). Uniform Bounds on Fibers. *Let $A \in S_{m+n}$ be a definable set, then there exists $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^m$, the set $A_x = \{y \in \mathbb{R}^n | (x, y) \in A\}$ has at most N connected components.*
(ii). Definable Cylindrical Cell Decomposition of $\mathbb{R}^n$. *There is a partition $\mathbb{R}^n = \coprod D_i$ into finitely many pairwise disjoint definable subsets D_i, called cells, with respect to a given finite collection $\{U_j\}_{j \in J}$ of definable sets. This leads to the definition of dimension* $\dim A$ *of each definable set $A \in \mathcal{S}$.*
(iii). Dimension is Well-Behaved. *Let $A \in S_n$ be nonempty, then* $\dim \mathrm{fr}(A) < \dim A$.

3. O-Minimal Chow's Theorem

Working over subsets of $\mathbb{C}^n$, we are thinking of subsets of $\mathbb{R}^{2n}$. We will fix an o-minimal structure $\mathcal{S}$ and we will call a set A definable in place of $\mathcal{S}$-definable for convenience. The goal of this section is to prove the following theorem by Peterzil and Starchenko [9], [10].

Theorem 2. O-Minimal Chow's Theorem. *Let $Y \subseteq \mathbb{C}^n$ be a closed analytic subvariety whose underlying set is definable in an o-minimal structure $\mathcal{S}$. Then Y is algebraic.*

The information of Y being analytic is local: By definition, for each $y \in Y$, there is an open neighborhood $U \subseteq \mathbb{C}^n$ of p such that $U \cap Y$ is common zero locus of holomorphic functions $f_1, ..., f_k$ on U. Comparatively, the conclusion of Y being algebraic is global: It states Y is common zero locus of polynomials $g_1, ..., g_l$ defined on the entire $\mathbb{C}^n$. So o-minimal Chow's theorem can be interpreted as the definability condition controls the global behavior of an analytic variety.

The classical Chow's theorem becomes a corollary:

Corollary. Chow's theorem. *Let $X \subseteq \mathbb{P}^n$ be a closed analytic subvariety, then X is algebraic.*

Proof. Choose any subspace $H := \mathbb{P}^{n-1} \subseteq \mathbb{P}^n$ and let $X^\circ = X \backslash H$, so X° is a closed analytic subvariety of $\mathbb{C}^n = \mathbb{P}^n \backslash H$.

On the other hand, X° is actually definable thanks to the notion of *analytic-geometric category* from [11]: Let $\mathscr{C}_{\text{an}}$ be the analytic-geometry category of subanalytic sets, so in particular, the inclusion map $i : \mathbb{C}^n \hookrightarrow \mathbb{P}^n$ is a $\mathscr{C}_{\text{an}}$-map, and the assumption implies $X \in \mathscr{C}_{\text{an}}(\mathbb{P}^n)$, so by property 1.5 in [11], $X^\circ = i^{-1}(X) \in \mathscr{C}_{\text{an}}(\mathbb{C}^n)$, in other words, X° is $\mathbb{R}_{\text{an}}$-definable. So by o-minimal Chow's theorem, X° is an algebraic subvariety of $\mathbb{C}^n$, i.e., X° is an affine variety.

Now by definition, the analytic closure of X° is X. On the other hand, the Zariski closure of X° is an algebraic variety $\bar{X}$ which contains X as closed analyic subvariety. By an argument from [8], page 58, two closures agree, so $\bar{X} = X$ and X is algebraic.

$\square$

Proof of Theorem 2. Our proof of theorem 2 majorly follows Bakker's notes [1], page 8-9. As a backup, we need the following lemma.

Lemma 1. *Any definable holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial.*

Proof. We prove it by induction. When $n = 1$, we claim that an entire definable function $f : \mathbb{C} \to \mathbb{C}$ is a polynomial. Otherwises, $z = 0$ is not a pole of $g(z) = f(1/z)$, so f has an essential singularity at ∞ (or equivalently for g at 0). By Big Picard theorem, almost all value $a \in \mathbb{C}$ are attained infinitely many times in an arbitrary small neighborhood of the essential singularity. Up to adding a constant, we can assume $a = 0$ is one of such value. On the other hand, f cannot have accumulated zero otherwise being a constant, it follows that $f^{-1}(0)$ is an infinite set and discrete, but this is not a definable set in $\mathbb{C}$, which contradicts to the assumption that f is definable.

Assume the lemma is true for $n - 1$. For a definable holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$, write $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ with z coordinate on $\mathbb{C}$ and w coordinates on $\mathbb{C}^{n-1}$. Take the Taylor expansion of $f(z, w)$ along the z axis:

$$f(z, w) = \sum_{k=0}^{\infty} g_k(w) z^k, \quad g_k(w) = \frac{1}{k!} \frac{\partial^k f}{\partial z^k}(0, w). \tag{1}$$

First of all, it is an exercise to show that the partial derivatives of a definable differntiable function are definable, so each $g_k(w)$ is definable, therefore a polynomial by the induction hypothesis.

Next, in order to show $f(z, w)$ is a polynomial, we need to show the sum in (1) is finite. Fix any $w \in \mathbb{C}^{n-1}$, it follows from the base step that $z \mapsto f_w(z) := f(z, w)$ is a complex polynomial. By projecting the graph $\Gamma(f) = \{(z, w, f(z, w)) \in \mathbb{C}^{n+1} | z \in \mathbb{C}, w \in \mathbb{C}^{n-1}\}$ to the last n-coordinates, the fiber at $(w, v) \in \mathbb{C}^{n-1} \times \mathbb{C}$ is

$$F_{w,v} = \{z : f(z, w) = v\}.$$

If $f_w \not\equiv 0$, the equality $|F_{w,v}| = \deg(f_w)$ holds when $f_w - v$ has no multiple roots; Otherwise f_w is a zero polynomial, so $F_{w,v}$ is either $\mathbb{C}$ or an empty set;

Now the Theorem 1(iii) applied to this projection says that the number of the connected components are uniformally bounded. In particular, $\deg(f_w)$ is uniformally bounded, say by $N \in \mathbb{N}$. It follows that $g_k \equiv 0$ for $k > N$, so $f(z, w) = \sum_{k=0}^{N} g_k(w) z^k$ is a polynomial. □

Step 1. A Generic Linear Projection is Proper. We begin to prove the Theorem 2. Assume $Y \subseteq \mathbb{C}^n$ is a closed analytic subvariety, and definable in some o-minimal structure (for example $\mathbb{R}_{\text{an}}$, $\mathbb{R}_{\text{an, exp}}$) as assumed in the theorem.

This section section is devoted to prove the following lemma.

Lemma 2. *There is a linear projection $\mathbb{C}^n \to \mathbb{C}^d$ whose restriction to Y*

$$\pi : Y \to \mathbb{C}^d$$

is proper, i.e., preimage of every compact set is compact.

The proof can be carried out inductively. The idea is that the "bad projection" comes from the projection center is close to locus of Y at infinity, but definablity guarantees the behavior of Y is tame at infinity, so the "bad projection" locus has codimension at least one. Intuitively, projecting $Y = \{(x, y) | xy = 1\}$ to x-axis is "bad" because Y is close to the y-axis asymtotically, but if we project to the line $x = y$, the map will be proper.

For those who are familiar with algebraic geometry, the idea of the proof is essentially the *Noether normalization theorem* for affine varieties.

Proof. We will first construct a projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ and then induct until $n = d$.

The set of all $(n-1)$-dimensional complex linear subspace of $\mathbb{C}^n$ is bijective to its orthogonal complement (using Hermitian inner product), which is bijective to the set of all 1-dimensional complex linear subspace of $\mathbb{C}^n$, or equivalently, the projective space $\mathbb{P}^{n-1}$.

Actually, we can add $\mathbb{P}^{n-1}$ to the boundary of $\mathbb{C}^n$ to form $\mathbb{P}^n$ by the following: If a line in $\mathbb{C}^{n+1}$ through origin intersect the affine subspace $\mathbb{H} = \mathbb{C}^n \times \{1\} \subseteq \mathbb{C}^{n+1}$, then it intersect at only one point, which gives the identification $\mathbb{H} \cong \mathbb{C}^n$; If a line does not intersect $\mathbb{H}$, then it is contained in the hyperplane $\mathbb{C}^n \times \{0\}$ and these lines form $\mathbb{P}^{n-1}$. These points are also referred to points at infinity. One can see how the two pieces are glued together by writing down coordinates ($n = 2$ case is the Hopf fibration).

In short,

Claim. *There is an inclusion $\mathbb{C}^n \subseteq \mathbb{P}^n$ such that the complement $\mathbb{P}^n \backslash \mathbb{C}^n$ parameterizes the set of linear projections $\mathbb{C}^n \to \mathbb{C}^{n-1}$.*

Explicitly, fix a point at infinity, say $[0, 0, ..., 1] \in \mathbb{P}^n$, denote $H = \{[*, ..., *, 0]\} \subset \mathbb{P}^n$ the projective subspace with last entry zero, one has the projection

$$\pi : \mathbb{P}^n \backslash \{[0, 0, ..., 1]\} \to H$$

$$[z_0, z_1, ..., z_n] \mapsto [z_0, z_1 ..., z_{n-1}],$$

whose restriction to $\mathbb{C}^n$ defines the projection

$$\pi|_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{C}^{n-1}. \tag{2}$$

Now, recall $Y \subseteq \mathbb{C}^n$ is a closed analytic subvariety, take its closure $\bar{Y}$ in $\mathbb{P}^n$. Since Y is definable, according to Theorem 1 (iii), the frontier $\mathrm{fr}(Y) = \bar{Y} \backslash Y \subseteq \mathbb{P}^{n-1}$ has real dimension at most $2d - 1$, so in particular it is not all of the $\mathbb{P}^{n-1}$. Then project from any point $y_\infty \in \mathbb{P}^{n-1} \backslash \mathrm{fr}(Y)$ as in (2) has bounded preimage over each bounded set of $\mathbb{C}^{n-1}$, so it follows that

Claim. *The restriction of projection from $y_\infty \in \mathbb{P}^{n-1} \backslash \partial Y$ to Y*

$$\pi|_Y : Y \to \mathbb{C}^{n-1}$$

is proper.

The famous *Remmert's proper mapping theorem* (see [6], p.34) states that if $f : M \to N$ is a holomorphic map between two complex manifolds, and A a analytic subvariety of M such that $f|_A$ is proper, then $f(A)$ is an analytic subvariety in N. According to this, $\pi|_Y(Y) \subseteq \mathbb{C}^{n-1}$ is an analytic subvariety. Moreover, it is closed since proper map preserves closedness; It still has dimension d since the map is finite; $\pi|_Y(Y)$ is still definable since it is restriction of linear projection. Now one can repeat the construction above until $n = d$. □

Step 2. Analytic Covering Map. Y is an closed analytic subvariety of $\mathbb{C}^n = \mathbb{C}^{n-d} \times \mathbb{C}^d$, with π the restriction to Y of the projection to the second factor and we can assume it is *proper* by the previous discussion. Also note that $\pi(Y)$ is a closed analytic subvariety of $\mathbb{C}^d$ with dimension d, so that is the whole $\mathbb{C}^d$. Therefore the projection

$$\pi : Y \to \mathbb{C}^d$$

is proper, finite and surjective.

There exist a closed proper analytic subvariety $Z \subseteq \mathbb{C}^d$ such that restriction of π to $Y \backslash \pi^{-1}(Z)$ is an analytic covering space. In other words, let $Y_0 := Y \cap \pi^{-1}(Z)$, denote

$U = Y \backslash Y_0$ and $V = \mathbb{C}^d \backslash Z$, then the map

$$\pi|_U : U \to V \tag{3}$$

is a local biholomorphism. Y_0 is a closed analytic subset of Y. Typically, Y_0 contains singular points of Y and those y where the tangent map $\pi_* : T_y Y \to T_{\pi(y)}\mathbb{C}^d$ is not surjective.

As a motivating example, take a polynomial f, project its graph $\{(z, f(z))\} \subseteq \mathbb{C}^2$ to the second coordinate, so branching locus is the set of values $v \in \mathbb{C}$ where $f - v$ has multiple roots, and ramification locus is those multiple roots on the graph.

We take Z the minimal one satisfying the conditions above, and call Z the *branching locus*. Additionally, there is no harm for us to assume Y is irreducible. If so, U will be connected covering space over V.

Step 3. Symmetric Functions are Extendable. We shall prove that Y is algebraic using the covering map (3). More precisely, we will find certain polynomials whose common zero locus defines Y. To better explain the idea, let me first assume that $d = n - 1$, that is, Y is a hypersurface of dimension d in $\mathbb{C}^{d+1}$. The general case will be discussed at the end.

Let m be the cardinality of the fiber of the $\pi|_U$. Let $x_0 \in V$, then there is an open neighborhood W of x_0 such that $\pi^{-1}(W)$ is biholomorphic to disjoint union of m-copies of W. Let

$$\phi_k : W \to \mathbb{C},\ i = 1, ..., m$$

be the holomorphic functions such that $x \mapsto (x, \phi_k(x)) \in W \times \mathbb{C} \subseteq \mathbb{C}^{d+1}$ are sections of projection $\pi^{-1}(W) \to W$. Therefore we can write $\pi^{-1}(W)$ as zero locus in $W \times \mathbb{C}$ of a single polynomial

$$\prod_{k=1}^{m} (y - \phi_k(x)) = 0. \tag{4}$$

Expand the polynomial (4), the equation becomes

$$y^m - \sigma_1(x) y^{m-1} + \sigma_2(x) y^{m-2} + \cdots + (-1)^m \sigma_m(x) = 0, \tag{5}$$

with coefficients

$$\sigma_1 = \sum_i \phi_i,\ \sigma_2 = \sum_{i<j} \phi_i \phi_j,\ ..., \sigma_m = \phi_1 \cdots \phi_m$$

are elementary symmetric functions on distinct m-points $\phi_1(x), ..., \phi_m(x)$ parameterized by x. These σ_k's are holomorphic and can be analytically continued to the entire V (in comparison ϕ_k cannot). So equation (5) for x varing in the entire V defines $U \subseteq \mathbb{C}^{d+1}$.

Recall that in complex analysis, *Riemann's extension theorem* says that if a holomorphic function f defined on $D - \{a\}$ is bounded around a, where $D \subseteq \mathbb{C}$ is an open subset, then f is holomorphically extendable to the entire D. The same statement holds in higher dimensions (see [5], p.80), with D replaced by an open set in $\mathbb{C}^n$ and $\{a\}$ by a (proper) analytic subset.

Now each σ_i is bounded in an neighborhood of branching locus $\pi(Y_0)$ as a result of π being proper, so by Riemann extension theorem implies that σ_k extends to a global holomophic function $\bar{\sigma}_k : \mathbb{C}^d \to \mathbb{C}$.

Definability. On the other hand, we should show $\bar{\sigma}_k$ is definable. Note that this will follow from definability of σ_k according to Proposition 2, so it suffices to show σ_k is definable. The idea is that, U is definable in the first place since it is the locus where the fiber size is maximal. To be more specific, V can be written as projection to $\mathbb{C}^d$ of a definable set

$$P_m := \big\{(x, y_1, ..., y_m) \in \mathbb{C}^d \times \mathbb{C}^m |\ y_i \in \pi^{-1}(x),\ i = 1, ..., m\}\backslash \bigcup_{i<j}\{y_i = y_j\}\big\},$$

so V is definable, and therefore $U = \pi^{-1}(V) \cap Y$ is definable.

To show that σ_k is definable, note that there is a $m!$-to-1 covering map

$$\tau_k : P_m \to V \times \mathbb{C},\ (x, y_1, ..., y_m) \mapsto (x, \sum_{l_1<\cdots<l_k} y_{l_1} \cdots y_{l_k})$$

which preserves projection to V. In other words, we have a commutative diagram:

$$\begin{tikzcd}
P_m \arrow[r, "\tau_k"] \arrow[d, "\pi"] & V \times \mathbb{C} \arrow[dl, "\pi"'] \\
V \arrow[ur, bend right, "\Gamma(\sigma_k)"'] &
\end{tikzcd}$$

Also note the image of τ_k coincides with graph of σ_k. This shows that σ_k is definable. It follows that $\bar{\sigma}_i$ is both definable and holomorphic, so is algebraic according to Lemma 1.

So we can consider the equation

$$G = y^m - \bar{\sigma}_1(x)y^{m-1} + \bar{\sigma}_2(x)y^{m-2} + \cdots + (-1)^m\bar{\sigma}_m(x) = 0$$

on $\mathbb{C}^{d+1}$.

Then one shows that $Y = \{G = 0\}$ is defined by a single polynomial, so it is algebraic. This finishes the proof when $n - 1 = d$.

General Situation. When $n - 2 \geqslant d$, the local section ϕ_i consists of $n - d$ holomorphic functions, so we need more equations to define $\pi^{-1}(W)$. One way to do it ([1], p.9) is to consider the symmetric product $\mathrm{Sym}^m\mathbb{C}^{n-d}$ which parameterizes unordered m-tuple of points in $\mathbb{C}^{n-d}$. It is an affine variety whose coordinate ring is the invariant subring of m-copy of $\mathbb{C}[t_1, ..., t_{n-d}]$ under permutation group S^m-action. Consider the map

$$F : V \to \mathrm{Sym}^m\mathbb{C}^{n-d},\ x \mapsto \pi^{-1}(x).$$

One can argue extendability of holomorphic map F, eventually, Y will be graph of the extended map $\bar{F}$ in $\mathbb{C}^d \times \mathrm{Sym}^m \mathbb{C}^{n-d}$. To extract polynomials explicitly, one need to find generators of coordinate ring of symmetric product.

Another approach ([4], p.44) gives more directly the set of defining equations. More explicitly, one consider the polynomial in $2n-d$ variables

$$P(x,y,w) = \langle w, y - \phi_1(x)\rangle \cdots \langle w, y - \phi_m(x)\rangle$$

where $\langle a, b\rangle := \sum_{i=1}^m a_i b_i$ and the author shows that the coefficients in the expansion $P(x,y,w) = \sum_{|I|=m} \eta_I(x,y) w^I$ gives the set of defining equation of U. Similarly their extensions will be definable and holomorphic, and cuts out Y in $\mathbb{C}^n$, so Y is algebraic. □

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