Rationality and Stable Rationality /$\mathbb{C}$

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Introduction: A smooth projective variety X is called rational if there is a birational map $X \dashrightarrow \mathbb{P}^n$ for $n = \dim X$. It is called unirational if there is a dominant rational map $\mathbb{P}^N \dashrightarrow X$. It is called stably rational if $X \times \mathbb{P}^r$ is rational for some r.

In $\dim(X) = 2$ case, these concepts are the same. But starting from $\dim(X) = 3$, the following implications are strict:

$$\text{rational} \Rightarrow \text{stably rational} \Rightarrow \text{unirational}.$$

Today we will: 1: Review Clemens-Griffiths' criterion on rationality for 3-folds, and show cubic 3-fold is unirational, but not rational.

2. Go through some stable birationality invariants, and sketch the proof of very general quartic double solid with ≤ 7 nodes is unirational but not stably rational.

Radu's remark: • Iskovskih and Manin showed that quartic 3-fold is not rational using birational automorphism group.

• Artin and Mumford showed Tor $H^3(X, \mathbb{Z})$ is a stably birational invariant and constructed certain nodal double solid which is not stably rational. (which we will use in this talk)

I. Clemens-Griffiths Criterion.

Let $X = \{f_3(x_0, x_1, \cdots x_4) = 0\} \subseteq \mathbb{P}^4$ be a (smooth) cubic 3-fold.

then its Hodge number in degree 3 is $H^3: \boxed{0 \mid 5 \mid 5 \mid 0}$

One way to look at it is through Griffiths residue: $H^{p, n-p}$ smooth hypersurface can be represented by residues of rational forms on projective space with poles along X to order p.

$$\mathrm{Res}\, \frac{Q\,\Omega}{f^p} \qquad \text{where } \Omega = \sum_i (-1)^i x_i\, dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}$$

Q homogeneous, $\deg Q + n+2 = p \cdot \deg(f)$

In our case. $\deg(f) = 3$, $n = 3$.

$\frac{Q\Omega}{f}$ is not possible. $\Rightarrow h^{3,0} = 0$.

$\frac{Q\Omega}{f^2} \Rightarrow Q$ is a linear form $\Rightarrow h^{2,1} = 5$.

Recall that $H^3(X, \mathbb{C}) = F^2H^3 \oplus \overline{F^2H^3}$, so the composite

$H^3(X, \mathbb{Z}) \to H^3(X, \mathbb{R}) \longrightarrow H^3(X, \mathbb{C}) \longrightarrow \overline{F^2H^3}$ is of maximal rank

$\cong$ as $\mathbb{R}$-vector space

so we define the intermediate Jacobian (Lecture 18 in Radu's class)

$$J(X) = \frac{\overline{F^2H^3(X, \mathbb{C})}}{H^3(X, \mathbb{Z})} \cong \frac{F^2H^3(X, \mathbb{C})^\vee}{H_3(X, \mathbb{Z})} \cong \frac{H^{2,1}(X)^\vee}{H_3(X, \mathbb{Z})}$$

Since $\langle \cdot, \cdot \rangle$ on $H^3(X, \mathbb{Z})$ is unimodular. $J(X)$ is principally polarized. So $J(X)$ is an abelian variety with theta divisor Θ with $h^0(\Theta) = 1$ $\quad$ $J(X)$ is p.p.a.v.

Let F be the variety of lines of X. i.e. $F \subseteq Gr(2,5)$ is the closed subvariety consisting of lines in $\mathbb{P}^4$ that are contained in X.

F is a smooth surface of general type, and $\mathrm{Alb}\, F \cong J(X)$

We have Abel-Jacobi map:

$$\varphi : F \times F \longrightarrow J(X)$$
$$(p, q) \longmapsto \int_{L_q}^{L_p} = \int_\Gamma$$

Γ = 3 chain such that $\partial \Gamma = L_p - L_q$

Theorem (Clemens–Griffiths, 72) Let X be a smooth projective 3-fold then X is rational $\Longrightarrow$ $J(X) \cong J(C_1) \oplus \cdots \oplus J(C_k)$ as p.p.a.v.

$C_1, \dots, C_k$ curves.

Proof: If $\mathbb{P}^3 \dashrightarrow X$ is birational, then Hironaka. there is a sequence of blowups $\tilde{P} = Y^m \to Y^{m-1} \to \cdots \to Y' \to \mathbb{P}^3$ along smooth centers. such that $\tilde{P} \to \mathbb{P}^3 \dashrightarrow X$ is a morphism

so we have

$$\begin{tikzcd} & \tilde{P} \arrow[dl, "\varphi"'] \arrow[dr, "\psi"] & \\ \mathbb{P}^3 \arrow[rr, dashed] & & X \end{tikzcd}$$

such that φ is a sequence of blowups and ψ is a birational morphism

Now smooth centers on 3-folds are points or curves

blowup a point p: $H^3(Bl_p Y) = H^3(Y)$.

blowup a curve C: $H^3(Bl_C Y) \cong H^3(Y) \oplus H^1(C)$ (reverse a sign on intersection pairing)

Apply to sequence $\tilde{P} \to \cdots \to \mathbb{P}^3$ and pass to Jacobian. we have

$$J(\tilde{P}) = J(C_1) \oplus J(C_2) \oplus \cdots \oplus J(C_\ell) \qquad C_1, \dots, C_\ell \text{ smooth curves}$$

Since ψ is a morphism. we have pullback.

$$\psi^*: H^3(X, \mathbb{Z}) \longrightarrow H^3(\tilde{P}, \mathbb{Z})$$

We have pushforward $\psi_*: H^3(\tilde{P}, \mathbb{Z}) \to H^3(X, \mathbb{Z})$

Since both maps respects Hodge structures. intersection pairings.

and $\psi_* \psi^* = Id$ due to $\deg(\psi) = 1$. we have ψ^* includes $H^3(X, \mathbb{Z})$ as a direct factor of $H^3(\tilde{P}, \mathbb{Z})$, so pass to Jacobian we have

$$J(X) \oplus A \cong J(\tilde{P}) \cong J(C_1) \oplus \cdots \oplus J(C_k).$$

Use the fact that decomposition of p.p. a.v. is unique, we conclude

$$J(X) \cong J(C_{i_1}) \oplus \cdots \oplus J(C_{i_p})$$

$\boxtimes$

Theorem (Clemens-Griffiths [72], Beauville [81]) Let X be a smooth cubic 3-fold. then X is irrational.

Sketch of the proof. By Clemens-Griffith criterion. we need to show $J(X)$ is not a product of Jacobian

Recall we have Abel-Jacobi map

$$\underset{\dim F = 2}{F \times F} \xrightarrow{\psi} \underset{\dim = 5}{J(X)} \qquad (p, q) \longmapsto \int_{L_q}^{L_p}$$

- ψ is generically 6-to-1. onto the theta divisor $\Theta \subseteq J(X)$
- ψ contracts the diagonal Δ to $0 \in \Theta$ which is an isolated singularity. and the tangent cone $TC_0\Theta$ is isomorphic to the affine cone of X. (This recovers Torelli theorem)
- Beauville showed Θ is irreducible and $\Theta^{sing} = \{0\}$. compare to the Riemann singularity theorem $\dim \Theta^{sing}_{J(C)} = \begin{cases} g-4 & C \text{ non-hyperelliptic} \\ g-3 & C \text{ hyperelliptic} \end{cases}$

So singularities of Θ of $\underbrace{J(C_1) \oplus \cdots \oplus J(C_k)}_{\dim = 5}$ is at least $\underline{1}$. $\boxtimes$

Remark: (1). In fact X is unirational: Fix $L \subseteq X$ a line.

Consider $\mathbb{P}(T_X|_L)$ which is rational variety, and

$\boxed{\text{rational}}$

$$f: \mathbb{P}(T_{X|L}) \overset{2:1}{\dashrightarrow} X$$
$$(x, v) \longmapsto L_{x,v} \cap X$$

where $L_{x,v}$ is the line through x in the direction v.

Conversely, take $u \in X$. $\mathbb{P} = \mathrm{Span}\langle u, L\rangle$ is a plane, which intersects X residually at a conic C. $C \cap L = 2$ points $= f^{-1}(u)$.

(2). Mumford showed that $(J(X), \Theta)$ is a Prym variety. Actually all p.p.a.v. of dim $=5$ is a Prym var.

(3). The Abel-Jacobi map lifts to the blowup:

$$([L_1, L_2]) \longmapsto \mathcal{O}_S(L_1 - L_2)$$

$S \subseteq X$ hyperplane section

$$\begin{array}{ccc} Bl_\Delta(F\times F) & \xrightarrow{\tilde{\varphi}} & Bl_0\,\Theta \\ \downarrow & & \downarrow \\ F\times F & \xrightarrow{\varphi} & \Theta \end{array}$$

(Z. 2021) showed that $Bl_\Delta(F\times F)$ is a branched double cover of Hilbert schemes of skew lines $H(X)$, which is an irreducible component of $\mathrm{Hilb}^{2n+2}(X)$.

$\cong Bl_\Delta \mathrm{Sym}^2 F$

(Altavilla-Petkovic-Rota. 19, Bayer et.al. 20) showed that $Bl_0\,\Theta$ is a Bridgeland stable moduli space. [with stable objects in $Ku(X)$]

$\tilde{\varphi}$ has certain modular interpretation.

(4). Let (A, Θ) be a p.p.a.v. of $\dim = g$

then $\Theta^{g-1}/(g-1)! \in H^2(A, \mathbb{Z})$ is called the

minimal class. When $A = J(C)$ is a Jacobian of a curve C,

minimal class is represented by $C \xrightarrow{\text{Abel-Jacobi}} J(C)$. Conversely

(Matsusaka's theorem) The minimal class is represented by an effective 1-cycle

$$\Updownarrow$$

$$(A, \Theta) \simeq J(C_1) \oplus \cdots \oplus J(C_k).$$

C_i = smooth curve

Therefore, for $J(X)$, where X = cubic 3-fold

then the minimal class is $\boxed{\text{not}}$ the class of a curve.

- Voisin showed that if X is a rationally-connected 3-fold.

  X is stably rational $\Longrightarrow$ minimal class of JX is algebraic

  ! not necessarily effective

Open Problem (Voisin): Is the minimal class of a cubic 3-fold algebraic? Is cubic 3-fold stably rational?

II. Classical stable birational invariants

Recall that X is stably birational to Y iff

$$X \times \mathbb{P}^n \text{ is birational to } Y \times \mathbb{P}^m$$ for some m, n

So typically a birational invariant is not a stably birational inv.

Here are some classical stably birational invariants (for smooth projective varieties):

- $\pi_1(X)$
- $H^0(X, (\Omega^1)^{\otimes k})$
- $\mathrm{Tor}\, H^3(X, \mathbb{Z})$: Artin-Mumford invariant which coincides with $Br(X)$, when X is rationally connected.

Let's prove the 3rd one is invariant under $X \underset{\text{sta. bir.}}{\sim} Y$

This relationship, on the set of all smooth projective varieties is generated by ① $X \sim X \times \mathbb{P}^r$ and ② $X \sim Y$ if X is birational to Y. For ①: By Kunneth formula.

$$\mathrm{Tor}\, H^3(X \times \mathbb{P}^r, \mathbb{Z}) \cong \mathrm{Tor}\, H^3(X, \mathbb{Z}) \oplus \underbrace{H^1(X, \mathbb{Z})}_{\text{torsionfree}}$$

By universal coefficient thm. $H^1(X, \mathbb{Z}) \cong \underbrace{\mathrm{Ext}^1(H_0(X, \mathbb{Z}), \mathbb{Z})}_{\text{torsionfree.}}$

For ②. if $X \dashrightarrow Y$ birational. by Hironaka.

$\tilde{X} \to \cdots \to X$ blowup along smooth centers. and

$\tilde{X} \longrightarrow Y$ is a morphism. But $\mathrm{Tor}\, H^3(Bl_Z X) \cong \mathrm{Tor}\, H^3(X) \oplus \underbrace{H^1(Z)}_{\text{torsionfree}}$

So $\mathrm{Tor}\, H^3(Y, \mathbb{Z}) \hookrightarrow \mathrm{Tor}\, H^3(\tilde{X}, \mathbb{Z}) \cong \mathrm{Tor}\, H^3(X, \mathbb{Z})$

We also have inclusion from the other direction, so isomorphism ☒

In particular, if X is stably rational, $\pi_1(X)=0$, $H^0(X,(\Omega^1)^{\otimes k})=0$, and $\mathrm{Tor}\,H^3(X,\mathbb{Z})=0$

Def: A quartic double solid is a variety which is 2:1 to $\mathbb{P}^3$ branched along a quartic surface

(Artin-Mumford, 72) constructed a double solid X_0 with 10 nodes, and showed its resolution of singularity $\tilde{X}_0$ has nontrivial $\mathrm{Tor}\,H^3(\tilde{X}_0,\mathbb{Z})$, so $\tilde{X}_0$ is not stably rational.

The construction can be done via the following:
Take a general 3-dim'l linear subsystem Π of $\mathbb{P}[H^0(\mathbb{P}^3,\mathcal{O}(2))^*] \cong \mathbb{P}^9$ parameterizing quadric surfaces in $\mathbb{P}^3$.

Then general member of Π is smooth, and there is a quartic surface $S \subseteq \Pi$ whose smooth points parameterize nodal quadrics, and $S^{sing} = \{10 \text{ points}\}$ parameterizes 

Consider the incidence variety:

$$\Gamma = \{(Q,L) \in \Pi \times Gr(2,4) \mid L \subseteq Q\}.$$

The first projection admits Stein factorization

$$\Gamma \xrightarrow{\mathbb{P}^1} X_0 \xrightarrow{2:1} \Pi,$$

where X_0 is 2:1 to Π branched along S, so X_0 has 10 nodes

(!) It is the fact that $\Gamma^{sm} \to X_0^{sm}$ admits no rational section, this $\mathbb{P}^1$-bundle defines a nontrivial element in

$$Br(X_0^{sm}) = \coprod_{r \in \mathbb{N}} Proj_r(X_0^{sm}) / \sim$$

Question (A): Is there a 3-fold which is not stably rational, and Artin-Mumford invariant vanish?

III. Voisin's stable birational invariants

1. Universal codimension 2 cycle

Let X be a rationally connected variety, then $H^{3,0}(X) \simeq 0$ and $J^3(X)$ is an abelian variety

Moreover, due to Bloch and Srinivas,

$$\Phi_X : CH^2(X)_{hom} \longrightarrow J^3(X)$$

is an isomorphism $\quad \sum n_i (E_i - C_i) \longmapsto \sum n_i \int_{C_i}^{E_i}$ $\quad C_i, E_i$: curves

It is also "regular" in the sense that for any smooth algebraic var B and $Z \in CH^2(B \times X)$ such that B_t is homologous to zero, $\forall t \in B$, then

$$\Phi_Z : B \longrightarrow J^3(X) \quad \text{is regular.}$$
$$t \longmapsto \Phi_X(B_t)$$

Remark: This is a stably birational property!

Question (B): let X be a smooth rationally connected variety. Does there exist a codimension 2 cycle $Z \in CH^2(J^3(X) \times X)$ such that Z_t, $t \in J^3(X)$, are homologous to zero, and the morphism

$$\Phi_Z : J^3(X) \to J^3(X)$$

is the identity? Remark: For codimension one cycles map to $Pic(X)$, this is true (can think of Poincaré bundle in curve case)

The following theorem answered both Question (A) and (B):

Main Theorem (Voisin, 2015) complementary of countably union of proper subvarities

(1) The desingularization of the very general quartic double solid with $k \le 7$ nodes in general position is not stably rational. (Moreover, Artin-Mumford invariant $= 0$)

(2) The desingularization of the very general quartic double solid with 7 nodes in general position does not admit universal codimension two cycle on $J^3(X) \times X$.

Why ≤ 7 (or 7) nodes? Given $k \le 7$ general points in $\mathbb{P}^3$, there is a linear space of dimension $34-4k$ of quartic homogeneous polynomials having zero differential at these k points. Thus there is an irreducible variety parameterizing quartic double solids with k nodes in general position. As for part (2). For smooth quartic double solid $h^{2,1}=10$. The number drop by 1 as appearance of each node (just like curve case). When there are 7 nodes, $J(\tilde{X})$ is p.p.a.v of dim $=3$, which is a Jacobian of a genus 3 curve, so the minimal class is algebraic.

The rest is devoted to the proof of Voisin's theorem. We need to introduce another invariant:

$\dim X = n$

<u>Definition</u>: Let X be a smooth projective variety.

(1) We say X has a Chow decomposition of diagonal if

$$\Delta_X = X\times\{x\} + Z \quad \text{in} \quad CH^n(X\times X)$$

where $x\in X$ is a point and Z is a cycle supported on $D\times X$, for some proper closed algebraic subset $D\subsetneq X$.

(2) We say X has a cohomological decomposition of diagonal if $[\Delta_X] = [X\times\{x\}]+[Z]$ in $H^{2n}(X\times X,\mathbb{Z})$ where Z is as above.

Remark: We may need to assume the integral Hodge conjecture on $X\times X$ to guarantee application to Künneth to $[\Delta_X]$ gives you cohomological DOD, and may also

Clearly: $\boxed{\text{Chow DOD} \Rightarrow \text{Cohomological DOD}}$

worry about the condition $Z \subseteq D \times X$.

example: $\mathbb{P}^n$ admits Chow DOD:

Recall $CH^*(\mathbb{P}^n \times \mathbb{P}^n)$ has basis $pr_1^* h^i \cdot pr_2^* h^j$

So $\Delta_{\mathbb{P}^n} = \sum_{i=0}^{n} pr_1^* h^i \cdot pr_2^* h^{n-i}$

$= \mathbb{P}^n \times \{x\} + \underbrace{\sum_{i=1}^{n} pr_1^* h^i \cdot pr_2^* h^{n-i}}_{\text{supported on } H \subseteq \mathbb{P}^n}$

Fact: existence of Chow (cohomological) DOD is a stably birational property of smooth proj var.

In particular, stably rational varieties admit DOD.

(A Degeneration thm on DOD) Let $\pi : X \to B$ be a flat projective morphism of relative dimension $n \geq 2$. where B is a smooth curve. Assume X_t smooth $t \neq 0$. and X_0 has at worst ordinary double points. Then

(1) If for general $t \in B$, X_t admits Chow DOD then the same is true for any smooth projective model $\tilde{X}_0$ of X_0.

(2) with additional assumption. that $H^{2*}(\tilde{X}_0, \mathbb{Z})$ is algebraic, same statement for cohomological DOD.

To prove Main Theorem (1). We degenerate the double solid with k nodes to the Artin-Mumford double solid. then use $\mathrm{Tor}\, H^3(\tilde{X}_0, \mathbb{Z}) \neq 0$ and Degeneration theorem.

For Main Theorem (2), we need to relate all obstruction to stable rationality: Voisin showed for rationally connected 3-folds, the LHS is equivalent to RHS:

admit Cohomological DOD $\Longleftrightarrow$

(1) $\mathrm{Tor}\, H^3(X, \mathbb{Z}) = 0$

(2) existence of universal codim $=2$ cycle on $J^3(X) \times X$

(3) The minimal class of $J(X)$ is algebraic

For a general quartic solid with 7 nodes,

LHS is false. (1), (3) are true. So (2) is false

Endrass   due to $\dim J^3(X) = 3$

☒.

So we proved the non-existence of the universal codim $=2$ cycle. However, Voisin said there need more understanding of this result.

Appendix 1

A remark on admitting cohomological DOD $\Rightarrow$ (ii):

$\dim X = 3$

If we have $[\Delta_X] = [X \times \{x\}] + [Z]$ in $H^6(X \times X, \mathbb{Z})$,

We can act on $H^3(X, \mathbb{Z})$ via correspondence (just like the correspondence on Chow group level that we learned in Lecture 19 of Radu's class). So $\alpha \in H^3(X, \mathbb{Z})$

$$\underbrace{[\Delta_X] \cdot \alpha}_{\alpha} = \underbrace{[X \times \{x\}] \cdot \alpha}_{0} + [Z] \cdot \alpha$$

So $\underline{\alpha = [Z] \cdot \alpha}$ $(*)$.

On the other hand $Z \subseteq \underset{\text{surface}}{D} \times \underset{\text{3-fold}}{X}$, take desingularization $\tilde{D} \to D$. Z lifts to $\tilde{Z} \in CH^2(\tilde{D} \times X)$, and let $j : \tilde{D} \to X$ be the composite. Then $(*)$ can be expressed as

$$\alpha = j_* \left[[\tilde{Z}]^* \alpha \right],$$

where $[\tilde{Z}]^* : H^3(X, \mathbb{Z}) \to H^1(\tilde{D}, \mathbb{Z})$

$j_* \quad H^1(\tilde{D}, \mathbb{Z}) \to H^3(X, \mathbb{Z})$

are both morphism of Hodge structures, so induces morphism b/w Jacobians.

This allows us to lift to codimension-one cycles on $\tilde{D}$ and consider morphism to $\mathrm{Pic}(\tilde{D})$, which should admit a universal codimension-one cycle as we previously remarked.

We have a diagram:

$$\begin{tikzcd}
CH^2(X)_{hom} \arrow[r, "Z^*"] \arrow[d, "\Phi_X"] \arrow[rr, bend left, "Id"] & CH^1(\tilde{D})_{hom} \arrow[r, "j_*"] \arrow[d, "\Phi_{\tilde{D}}"] & CH^2(X)_{hom} \arrow[d, "\Phi_X"] \\
J^3(X) \arrow[r, "{[\tilde{Z}]^*}"] \arrow[rr, bend right, "Id"'] & J^1(\tilde{D}) \arrow[r, "j_*"] & J^3(X)
\end{tikzcd}$$

Let W be the universal cycle on $J^1(\tilde{D}) \times \tilde{D}$, one get codim $=2$ cycle on $J^3(X) \times X$ via

$$\left(Id_{J^3(X)}, j\right)_* \left(\left([\tilde{Z}]^*, Id_{\tilde{D}}\right)^* W\right)$$

☒

Appendix 2 An explanation on why the image of

$\varphi : F \times F \longrightarrow JX$ where $X =$ cubic 3-fold

is the theta divisor $\Theta \subseteq JX$:

Step 1: $Im(\varphi)$ is a divisor

There is a factorization

$$\begin{tikzcd}
F \arrow[r, "\varphi"] \arrow[d, "alb"] & JX \\
Alb(F) \arrow[ur, "\cong"'] &
\end{tikzcd}$$

$\varphi : p \longmapsto \int_{L_{p_0}}^{L_p}$ is an embedding and the tangent map together with projection

$$TF \longrightarrow TJX \simeq JX \times T_0 JX \longrightarrow T_0 JX$$

has projectivization

$$\mathbb{P}TF \longrightarrow \mathbb{P}T_0 JX \cong \mathbb{P}^4$$

identified with the second projection of incidence variety

$$I = \{(t, x) \in F \times X \mid x \in L_t\}$$

This implies $\psi : (s, t) \longmapsto \int_{L_t}^{L_s}$ has tangent map being of maximal rank. So $\operatorname{Im}\psi$ has $\dim = 4$

<u>step 2 $\operatorname{Im}(\psi)$ has Poincare dual class being $[\Theta]$, the principal polarization class of $J(X)$</u>

This reduces to show $F \hookrightarrow J(X)$ has Poincare dual class $[\Theta]^3/3!$. This requires some deeper analysis on topology on F. In Clemens-Griffiths, they degenerate X to a cubic 3-fold with an ordinary node, and the corresponding Fano surface F_0 has a double line. One understand topology of F via degenerating to F_0 ...

Alternatively, [Beauville. 81] has an interpretation of

$$F \times F \longrightarrow JX$$

in terms of Prym varieties, where the theta divisor is characterized as set of divisors of certain degree with $h^0 > 0$. One reduce to an algebra problem. This can replace step 2, but not step 1.

$\boxtimes$

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