RESEARCH STATEMENT

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The Abel-Jacobi map for a compact Riemann surface C is a morphism from the symmetric product of C to its Jacobian $J(C)$

$$Sym^k C \to J(C),\ (p_1, \dots, p_k) \mapsto \big(\omega \mapsto \sum_i \int_{p_0}^{p_i} \omega\big) \tag{1}$$

that integrates holomorphic 1-forms along paths joining a fixed point p_0 to p_i. It is used to prove the Lefschetz $(1,1)$-theorem, more commonly known as Hodge conjecture for algebraic surfaces, which states any integral $(1,1)$ class on a smooth projective complex surface S is represented by an integral linear combination of algebraic curves in S.

Griffiths generalized Abel-Jacobi map to higher dimensions. For example, when X is a smooth cubic hypersurface in the projective space $\mathbb{P}^4$, Abel-Jacobi map sends a pair of projective lines on X to a complex torus, called the intermediate Jacobi of X. It is used by Zucker to prove Hodge conjecture of cubic fourfolds.

My research is to study various geometric implications of the Abel-Jacobi map and its relation to problems in Hodge theory. On the other hand, we want to study a generalization of Abel-Jacobi map to vanishing cycles, called the topological Abel-Jacobi map.

1. Hodge classes on product of varieties

1.1. Hodge Conjecture for Theta Divisors.

In [4], it is proved that the Hodge conjecture is true for the theta divisor of a very general principally polarized abelian fivefold. Note such a theta divisor is smooth since the abelian variety is general. However, there are (at least) two type of special abelian fivefolds that are potentially interesting:

(1) Jacobian $J(C)$ of a genus five curve C.
(2) Intermediate Jacobian $J(X)$ of a cubic threefold X.

The theta divisors are singular in both cases, denoted as Θ_C and Θ_X. There is a version of Hodge conjecture for singular varieties [6, A.8]. Essentially it asks for the Hodge conjecture on a resolution of a singular variety but is independent of choosing a resolution. So it would be interesting to ask Hodge conjecture for these theta divisors.

Θ_C is singular in dimension at least one and can be resolved using Abel-Jacobi map (1). Then by pulling back to a finite cover, the question is about Hodge conjecture on product of curves C^4. Θ_X is singular at a single point due to Beauville and can be resolved by blowing it up. Using Abel-Jacobi map [1], it lifts to a blow-up of $F \times F$ along the diagonal, where F is a surface of general type parametrizing projective lines on X. So the question reduces to

Question 1.1. *Does the Hodge conjecture hold for C^4 and $F \times F$?*

I plan first to study Question 1.1 for general C and X and identify potential Hodge classes. This may need Mumford-Tate groups. When a curve C is special, the Hodge conjecture for

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Date: Mar 31, 2023.

product of curves can be widely open in general. However, one may relax Question 1.1 and ask for only a subset of Hodge classes of interest. This is related to the next project.

1.2. **Hodge Conjecture for CM Abelian Varieties.** Chad Schoen [8] studied a specific Hodge substructure U of type $(2,2)$ on product of a genus seven curve C_7 with automorphism of order three and found explicit representing algebraic cycles. Then by Abel-Jacobi map (1), he proved the Hodge conjecture for the associated Prym variety, which leads to proof of all abelian fourfolds with CM structure $\mathbb{Q}(\sqrt{-3})$.

The same strategy can be adapted to prove Hodge conjecture of abelian fourfold with CM structure $\mathbb{Q}(i)$. This is achieved by van Geeman in [9]. He asked if one can show Hodge conjecture for abelian sixfold with CM structure $\mathbb{Q}(i)$.

In an ongoing project with Deepam Patel, we aim to solve this problem. This requires understanding if the moduli of Prym varieties of interest dominate the moduli of abelian varieties with the given CM structure. We also want to apply Schoen's method to study Hodge conjecture of "Prym varieties" associated with intermediate Jacobian of Fano threefolds (e.g., cubic threefolds) with automorphism.

2. Abel-Jacobi Map and Geometry

2.1. **Topological Abel-Jacobi Map.** Let $X \subseteq \mathbb{P}^N$ be a smooth projective variety of dimension $2n-1$, and Y a smooth hyperplane section. Zhao defined *topological Abel-Jacobi map* [12] that sends vanishing cycles on Y to the (primitive) intermediate Jacobian of X.

We showed that Zhao's construction coincides with another definition of topological Abel-Jacobi map using $\mathbb{R}$-split mixed Hodge structures proposed by Schnell [11]. We also studied the map explicitly for cubic threefold [10]. For example, we studied a component of the underlying analytic space of the local system on vanishing cohomology and its compactifications. We found when the topological Abel-Jacobi map extend to the compactifications and gave geometric reasons. In the future, we want to study:

Question 2.1. *How to describe the image of the topological Abel-Jacobi map for hypersurface of $\mathbb{P}^4$ with degree at least four?*

In particular, we want to study the topological Abel-Jacobi map in families and how it behaves on the locus of Hodge classes.

2.2. **Two Equivalence Relations on Algebraic Cycles.** Let X be a smooth projective variety of dimension $2n-1$ over $\mathbb{C}$. Then the Abel-Jacobi image of the group of algebraic cycles that are algebraically trivial in the intermediate Jacobian is an abelian variety, denoted as J_a. Its tangent space is isomorphic as (a real vector space) to $H^{2n-1}_a(X,\mathbb{R})$, a subspace of $H^{2n-1}(X,\mathbb{R})$. In an ongoing project with Fumiaki Suzuki, we want to study

Question 2.2. *Is the restriction of the intersection pairing $(\cdot,\cdot)_{H^{2n-1}(X,\mathbb{R})}$ to $H^{2n-1}_a(X,\mathbb{R})$ nondegenerate?*

In fact, if the statement is true, then a conjecture of Griffiths [3, p.17, Problem B] predicts that the Abel-Jacobi equivalence and incidence equivalence on algebraic cycles only differs by a finite index will hold. Murre [7] answered the question positively for $n = 2$ ($n = 1$ is trivial). From the known examples (cubic threefold and some other Fano varieties), the two equivalence relations coincide. So we also want to understand (for codimension two cycles) if the two equivalence relations are actually the same.

2.3. **Degree of Gauss Map.** For any principally polarized abelian variety A of dimension g, there is a rational map from its theta divisor $\mathcal{G} : \Theta_A \dashrightarrow (\mathbb{P}^{g-1})^*$ called the Gauss map. It is generically finite, and the degree is bounded above by $g!$. The difference $\delta = g! - \deg(\mathcal{G})$ measures the singularities of Θ_A. δ is computed explicitly for theta divisor of the Jacobian of curve or intermediate Jacobian of a cubic threefold. In general, [2] gives a lower bound of δ, but it is not clear if the bound is "sharp." In an ongoing project with Wenbo Niu, we want to understand if the bound can be improved using resolution of singularities.

2.4. **Hyperplane sections of cubic threefold.** Let X be a general cubic threefold. Then by taking hyperplane sections, there is a generically finite rational map

$$\phi : (\mathbb{P}^4)^* \dashrightarrow \bar{M}_{cubic}, \tag{2}$$

that dominate the moduli space of cubic surfaces. In an ongoing project with Lisa Marquand, we would like to answer:

Question 2.3. *What is the degree of ϕ?*

As a consequence of a result of Kazaryan [5], the stratum of the dual variety X^* is known. Consequently, ϕ is regular except on a curve Γ whose general point parameterizes cubic surface with an A_3 singularity. Our method is to construct certain blow-up along these strata so that ϕ extends to a regular morphism. Then the problem reduces to compute intersection numbers among exceptional divisors of the resolution as well as the strict transform of X^*. Currently, we can describe the geometry of the blow-up ε and compute all the intersection numbers that are local in nature. Others seem to be related to the geometry of the curve Γ in the dual variety, which is the next thing that we want to understand.

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