DEPARTMENT OF MATHEMATICS

OSU CYCLE

# Mathematical Explanation of Analemma

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## Abstract

An analemma is a diagram showing the position of the Sun in the sky as seen from a fixed location on Earth at the same time in a day, the diagram will resemble a figure eight. Our project attempts to provide a mathematical understanding of why such a phenomenon exists and establish a general formula. We found the cause of the diagram formation using spherical geometry and calculus of vector-valued functions and established a general formula using differential equation and coordinate transformation. We end with a general formula for the Analemma depending on the observer's latitude and local time and an easy-to-understand model.

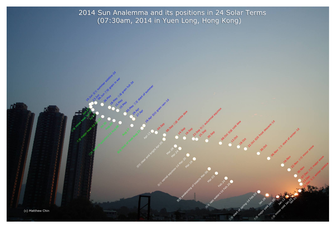


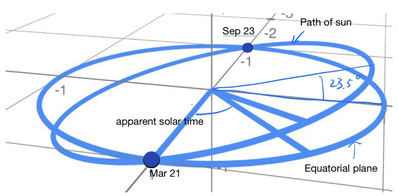

Figure 1:Analemma observed from Hong Kong at 7:30 am[1]

## Principle of formation

An analemma is a figure composed of the equation of time (the north-south component) and the declination of the sun (the east-west component) [2]. The equation of time describes the discrepancy between apparent solar time (time by the sun in the sky) and mean solar time (time on the clock) [3]. Due to fact that earth's orbit is an ellipse, apparent time becomes non-uniform variation, $\mathrm{e}$ is eccentricity:

$$\mathrm{p(t)} = \mathrm{t} + 2\mathrm{e} \cdot \sin(\mathrm{t} + 104\mathrm{days}/182.5\mathrm{days})$$

The other cause of the discrepancy is the tilt from the earth's orbit around the sun from the equator:




The apparent solar time is the projection to the sun traveling along the great circle tilted 23.5 degrees from the equator. Mean solar time is changing uniformly, which equals to $\mathrm{t}$. The angle from the projection will differ from $\mathrm{t}$a little bit, and we define $\mathrm{w(t)}$ to be their difference.

## Analemma on Celestial sphere

The celestial sphere is an abstract sphere that has an arbitrarily large radius and is concentric to the earth. On the celestial sphere, an image of analemmas for every 2 hours in a day are shown below.

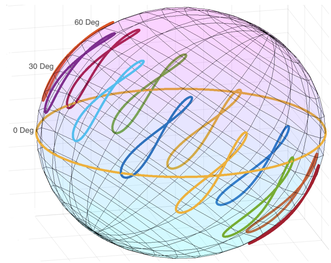



Figure 2:Analemma on Celestial sphere

The great circle in the middle of the ball indicates the horizon of the observer. Observer is at the center of the sphere. (Portions of) analemmas above the horizon are visible, while those beneath are invisible.

## General formula

We describe analemma with Azimuthal and Altitude (horizontal and vertical angles from the observer. By our general formula, an analemma observed at noon local time through a year on a fixed place on the earth can be calculated. The latitude of the observer is $\mathrm{L}$ (°). $\mathrm{d}$ is the number of the days past since Mar 21 (modulo 365 when necessary).

Convert time to angle:

$$\mathrm{t} = 2\pi(\mathrm{d}/365)$$

Apparent solar time:

$$\mathrm{w(t)} = \mathrm{Arccos}\frac{\cos(\mathrm{t})}{\sqrt{\cos^2(\mathrm{t}) + \sin^2(\mathrm{t})\cos^2(23.5°)}}$$
$$\mathrm{a(t)} = \mathrm{w(p(t))}$$

Analemma at noon:

$$\mathrm{Altitude} = \arcsin((\sin(\mathrm{t})\sin(23.5°))) + 90° - \mathrm{L}$$
$$\mathrm{Azimuth} = \mathrm{a(t)} - \mathrm{t}$$

Altitude is the vertical angle from the observer, It is actually the vertical variable change of the sun's path in the figure in principle of formation. Azimuth is the horizontal angle from the observer, this equals the equation of time, where $\mathrm{a(t)}$ is apparent solar time and $\mathrm{t}$ is mean solar time.

## Analemma Model by observing from one location on surface of earth

The figure below is the analemma at 12:00 noon at Greenwich Observatory (latitude 51.4791° north, longitude 0°) based on our formula and from astronomical observation.

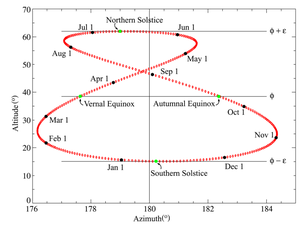



Figure 3:Analemma as seen at noon from the Observatory Greenwich, UK [4]

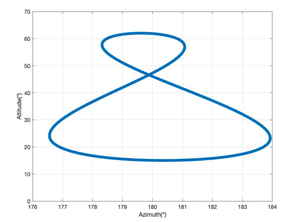



Figure 4:Analemma generated according to the general formula

## Discussion

In our calculation, we treat the sun as a point at infinity and express it in degrees. When calculating the effect of elliptical orbits on apparent time, we keep the first-order term and ignore the higher-order terms in eccentricity $\mathrm{e}$, making our result a relative approximation. Since the position of the sun varies in a day, an analemma will tilt at certain angle if observed at a time different from 12 pm. For example, if observer is in Hong Kong at 7:30 am as shown in Figure 1, through vector operation, the angle of inclination from the vertical is about 66.1°, compared to the measured from the photo is 66.7°.

## References

[1] Analemma 2014, from Hong Kong: Today's image. EarthSky. (2014, December 27). Retrieved April 7, 2022, from https://earthsky.org/todays-image/analemma-and-chinese-solar-terms-2014/

[2] Sawyer, Frederick (June 1994). "Of Analemmas, Mean Time and the Analemmatic Sundial - Part 1". Bulletin of the British Sundial Society. 6 (2): 2–6.

[3] Meeus, J (1997). Mathematical Astronomy Morsels. Richmond, Virginia: Willman-Bell.

[4] NASA. (n.d.). Horizons system. NASA. Retrieved April 7, 2022, from https://ssd.jpl.nasa.gov/horizons/