The Riemann Mapping Theorem

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The proof of the Riemann Mapping Theorem is a collection of propositions from Steven R. Bell’s MA530 class notes in Spring 2010. The familiarity with the Maximum Principle and the Schwarz lemma is assumed.

Lemma 1 (Log and Root function). Let $f$ be analytic and its domain $\Omega$ simply connected. If $f$ is non-vanishing, then for any $N \in \mathbb{Z}_+$, there is an analytic function $h$ such that $h^N = f$.

Proof. Since $f$ is non-vanishing, there is no problem to write $\frac{f'}{f}$ and it is analytic. Fix a point $A$ in $\Omega$ and let $z \in \Omega$. Because $\Omega$ is path connected, there is at least one path $\gamma_A$ in $\Omega$. So we can define $G(z) := \int_{\gamma_A} \frac{f'(\omega)}{f(\omega)} d\omega$. But $\Omega$ is simply connected and so $G$ is independent of paths and thus well-defined. We will also see that $G$ is analytic. Let $z_0 \in \Omega$. Since $\Omega$ is open, some disk $D_r(z_0)$ is contained in $\Omega$ and thus for any $z \in D_r(z_0)$, the straight line $L_{z_0}$ lies in $D_r(z_0)$ and thus in $\Omega$. By simple connectedness, $F(z) - F(z_0) = \int_{\gamma_{z_0}} \frac{f'(\omega)}{f(\omega)} d\omega = \int_{L_{z_0}} \frac{f'(\omega)}{f(\omega)} d\omega$. Therefore,

$$\lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = \lim_{z \to z_0} \int_{L_{z_0}} \frac{f'(\omega)}{f(\omega)} - \frac{f'(z_0)}{f(z_0)} d\omega \leq \lim_{z \to z_0} \frac{1}{z - z_0} \int_{L_{z_0}} \left| \frac{f'(\omega)}{f(\omega)} - \frac{f'(z_0)}{f(z_0)} \right| d\omega \leq \lim_{z \to z_0} \frac{\max_{\omega \in L_{z_0}} \left| \frac{f'(\omega)}{f(\omega)} - \frac{f'(z_0)}{f(z_0)} \right|}{z - z_0} = 0,$$

because $\frac{f'}{f}$ is continuous. This shows that $G$ is analytic. Notice that $(f e^{-G})' = f' e^{-G} - fG' e^{-G} = f'e^{-G} - f \frac{f'}{f} e^{-G} = 0$. Thus, $f e^{-G}$ is a constant $c$. So $f = ce^{-G} = e^{\ln c + G}$. Let $g := \ln c + G$. So $g$ is analytic and $f = e^g$. Hence, $h := e^{\frac{g}{N}}$ is analytic and we have $h^N = f$, as desired.

Corollary 2 (1-1 $\Rightarrow$ nonvanishing derivative). Let $f$ be analytic. If $f$ is one-to-one, then $f'$ is non-vanishing.

Proof. Let $z_0$ be in the domain of $f$ and let $\omega_0 = f(z_0)$. Then $f - \omega_0$ has $z_0$ as a zero. Notice that $f - \omega_0 \not\equiv 0$ since $f$ is one-to-one. Also because $f$ is analytic, $z_0$ is an isolated zero of $f(z) - \omega_0$. Hence, $f(z) - \omega_0 = (z - a)^N H(z)$ for some $N \in \mathbb{Z}_+$ and $H$ is analytic and $H$ is nonvanishing on some disk centered at $z_0$. By the previous lemma, there is analytic $h$ such that $H = h^N$ (we know $h(z_0) \neq 0$). Thus, $f(z) - \omega_0 = [(z - z_0)h(z)]^N$. Let $g(z) = (z - z_0)h(z)$. Then $f(z) - \omega_0 = g^N(z)$. Because $g'(z_0) = h(z_0) \neq 0$, by complex inverse function theorem, $g$ is locally one-to-one. Thus, $f$ is locally $N$-to-one. But $f$ is one-to-one. This forces $N = 1$. Thus, $f(z) = w_0 + g(z)$ and we have $f'(z_0) = g'(z_0) \neq 0$, as desired.

Lemma 3 (Rouché). Suppose $f$ and $g$ are analytic on $D_R(a)$ and $0 < r < R$. If $|f(z) - g(z)| < |g(z)|$ on $C_r(a)$, then $f$ and $g$ have the same number of zeros inside $C_r(a)$.
Proof. Since $|f(z) - g(z)| < |g(z)|$ on $C_r(a)$, $f$ and $g$ are both non-vanishing on $C_r(a)$. Hence, by the argument principle $\frac{1}{2\pi i} \int_{C_r(a)} \frac{f'(z)}{f(z)}dz$ and $\frac{1}{2\pi i} \int_{C_r(a)} \frac{g'(z)}{g(z)}dz$ are the numbers of zeros of $f$ and $g$ inside $C_r(a)$, respectively. Notice that

$$\frac{1}{2\pi i} \int_{C_r(a)} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i} \int_{C_r(a)} f'(z) - \frac{g'(z)}{g(z)}dz,$$

which is the difference of the numbers of zeros of $f$ and $g$ inside $C_r(a)$. However, from $|f(z) - g(z)| < |g(z)|$ on $C_r(a)$ we know that $\frac{|f(z)|}{|g(z)|} - 1 < 1$. This means $\{z \in \mathbb{C} | \Re z \leq 0, \Im z = 0\}$ can be made a branch cut for $\Log f$. Therefore, $\frac{1}{2\pi i} \int_{C_r(a)} \frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i} \int_{C_r(a)} \frac{d}{dz} \Log f(z)dz = 0$. Thus, $f$ and $g$ have the same number of zeros inside $C_r(a)$.

\[\square\]

Corollary 4 (Hurwicz No.1). Suppose the sequence of analytic functions $\langle f_n \rangle$ on a domain $\Omega$ is nonvanishing. If $f_n \to f$ uniformly on compacts of $\Omega$, then either $f$ is non-vanishing on $\Omega$, or $f \equiv 0$ on $\Omega$ (e.g., $z^n$ on $D_1(0)\setminus\{0\}$).

Proof. If $f \equiv 0$, then we are done. Suppose $f \not\equiv 0$. So there is $z_0 \in \Omega$ such that $f(z_0) = 0$. Because $f_n$ are analytic and the convergence is uniform on compacts, $f$ is analytic (by Morera’s theorem + uniform limit of continuous functions is continuous + change of integral and limit signs by uniform convergence).

Thus, $z_0$ is an isolated zero (this can be seen by the continuity of $h$ where $f(z) = (z - z_0)^N h(z)$ and $h(z_0) \neq 0$. Thus, there is $r > 0$ such that $f$ is not zero on $D_r(z_0)\setminus\{z_0\}$ and $\overline{D_r(z_0)} \subset \Omega$. Because $f$ is analytic, $f$ is continuous. Thus, $|f|$ is continuous. Since we also know that $C_r(z_0)$ is compact and $f$ is nonzero on it, $\min_{C_r(z_0)} |f|$ exists and is nonzero. Because $f_n$ converges uniformly on compacts, there is $n \in \mathbb{Z}_+$ (actually, for all $n$ greater than a particular $N \in \mathbb{Z}_+$) such that $|f_n(z) - f(z)| < \min_{C_r(z_0)} |f|$ on $C_r(z_0)$. By Rouche’s theorem, $f_n$ and $f$ have the same number of zeros inside $C_r(z_0)$. But $f_n$ has none and $f$ has one. This is a contradiction. Consequently, $f$ is non-vanishing.

\[\square\]

Corollary 5 (Hurwicz No.2). Suppose $f_n$ are univalent (one-to-one and analytic) on a domain $\Omega$ which converges uniformly on compacts to $f$. Then either $f$ is univalent, or $f$ is constant on $\Omega$.

Proof. If $f \equiv C$, then we are done. Suppose $f \not\equiv C$. For the sake of contradiction, assume that there are $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$ such that $f(z_1) = f(z_2) = w$. Define functions $F_n := f_n - w$ and $F := f - w$. So $F(z_1) = F(z_2) = 0$ and $F$ is a non-constant analytic function. Repeating the proof of Hurwicz No.1, we see that there are two discs $D_{r_1}(z_1)$ and $D_{r_2}(z_2)$ with non-intersecting closures such that there is $n \in \mathbb{Z}_+$ (any $n$ greater than $N_1$ and $N_2$ would do it) such that $F_n$ and $F$ have the same number of zeros in $D_{r_1}(z_1)$ and $D_{r_2}(z_2)$, respectively. Hence, there are two distinct points making $F_n$ zero. Therefore, $f_n$ is not one-to-one, which is a contradiction. This shows $f$ must be one-to-one.

\[\square\]

Lemma 6 (Montel). Suppose $f_n$’s are analytic on a domain $\Omega$. If $f_n$’s are uniformly bounded on compact subsets of $\Omega$, then there is a subsequence $\langle f_{n_k} \rangle$ which converges uniformly on compact subsets to a analytic function $f$.

Proof. If we can show that $f_n$’s are equicontinuous, then since we also know $\Omega$ is separable and at each point (a one point set is compact) of its skeleton subset in $\Omega \{f_n\}$ is bounded, by Ascoli-Arzelà’s theorem (see the note in the Real Analysis section of the webpage), then there is a subsequence $\langle f_{n_k} \rangle$ which converges uniformly on compact subsets to a function $f$. We also know previously that $f$ is analytic
(See the hint in the proof of Hurwicz No.1).

Let \( z_0 \in \Omega \). Let \( D_r(z_0) \subset \Omega \) since \( \Omega \) is open. Because \( f_n \)'s are uniformly bounded on compacts, we may let \( M = \max_{D_r(z_0)} |f_n| \). Recall that \( f(z) = \frac{1}{2\pi i} \int_{C_{2r}(z_0)} \frac{f(\omega)}{\omega - z} d\omega \). If \( z \in D_r(z_0) \), then

\[
|f_n(z) - f_n(z_0)| = \frac{1}{2\pi} |z - z_0| \int_{C_{2r}(z_0)} \frac{f_n(\omega)}{(\omega - z)(\omega - z_0)} d\omega \leq \frac{1}{2\pi} |z - z_0| \frac{M}{(2r - r)^2} 2\pi(2r) = |z - z_0| \frac{2M}{r}.
\]

This shows \( f_n \)'s are quicontinuous on \( \Omega \).

\[\square\]

**Theorem 7** (Riemann Mapping Theorem). Let \( \Omega \) be a simply connected domain and \( \Omega \neq \mathbb{C} \). Then there is a bijective analytic function \( f : \Omega \rightarrow D_1(0) \).

**Proof.** Let \( F \) be the family of analytic functions mapping \( \Omega \) injectively into \( D_1(0) \) (we didn’t say the functions are surjective). We first claim that \( F \neq \emptyset \). To see it, we consider two cases. If \( \Omega \) is bounded, then \( f(z) = \frac{r}{z} \) is in \( F \) where \( R \) is the bound of \( |f| \). If \( \Omega \) is unbounded, then since \( \Omega \) is not \( \mathbb{C} \), there is \( \omega_0 \in \mathbb{C} \backslash \Omega \). Thus, the function \( g(z) := z - \omega_0 \) on \( \Omega \) is nonvanishing and analytic on simply connected \( \Omega \). Thus, by Lemma 1, there is an analytic function \( G \) on \( \Omega \) such that \( G^2(z) = g(z) \). Let \( G(z_1) = G(z_2) \), then \( G^2(z_1) = G^2(z_2) \), from where we get \( z_1 = z_2 \). So \( G \) is one-to-one. Because the proper subset \( \Omega \) of \( \mathbb{C} \) is simply connected and \( G \) is the square root function of \( g \), \( G(\Omega) \) misses some open disk \( D_t(a) \). Hence, if we denote \( H \) by \( H(z) = \frac{r}{z-a} \) which is one-to-one and analytic, \( F \circ G \) maps \( \Omega \) one-to-one into \( D_1(0) \).

For any \( f \in F \), because it is one-to-one, by Corollary 2, \( f' \) is non-vanishing. Let \( a \in \Omega \). Then, \( M := \sup_{f \in F} |f'(a)| \neq 0 \). On the other hand, by Cauchy’s estimates, \( |f'(a)| \leq \frac{\max_{\Omega} |f|}{1 - |a|} \leq \frac{1}{r} \). But \( r \leq \text{dist}(a, b\Omega) \). Hence, \( M \leq \frac{1}{\text{dist}(a, b\Omega)} \).

Let \( f_n \) be in \( F \) such that \( |f_n'(a)| \not\leq M \). Because \( f_n \) are uniformly bounded by 1, by Lemma 6, there is a subsequence \( f_{n_k} \) which converges uniformly on compact subsets of \( \Omega \) to an analytic function \( f \) on \( \Omega \). This is the mapping that we want.

Firstly, because \( f_{n_k} \rightarrow f \), \( f_{n_k}'(a) \rightarrow f'(a) \). Thus, \( |f_{n_k}'(a)| \rightarrow |f'(a)| \). Hence, \( |f'(a)| \neq 0 \). This shows \( f \neq C \). Therefore, since \( f_{n_k} \) are one-to-one, by Lemma 5, \( f \) is one-to-one.

Secondly, since \( f_{n_k} \rightarrow f \) and \( f_{n_k} : \Omega \rightarrow D_1(0) \), \( f : \Omega \rightarrow \overline{D_1(0)} \). But by the Maximum Principle, if \( |f| \) takes 1 which is a maximum at some point inside \( \Omega \), \( f \) would be constant, which violates that \( f'(a) \neq 0 \). Therefore, \( f \) maps \( \Omega \) into \( D_1(0) \).

Finally, we only need to show \( f \) is surjective. For the sake of contradiction, suppose there is \( \omega_0 \in D_1(0) \) such that \( \omega_0 \notin f(\Omega) \). Let \( \varphi_{-\omega_0}(z) = \frac{1}{z - \omega_0} \), then \( \varphi_{-\omega_0} \circ f \) is nonvanishing on simply connected \( \Omega \). Thus, by Lemma 1, there is analytic \( g \) on \( \Omega \) such that \( g^2 \equiv \varphi_{-\omega_0} \circ f \) and it is not hard to see that \( g \) is one-to-one since \( \varphi_{-\omega_0} \) and \( f \) are one-to-one. Denote \( b := g(a) \) and define \( h := \varphi_{-b} \circ g \). Then \( h \in F \). If we define \( F := \varphi_{-b}^{-1} \circ h \circ \varphi_{-b} \), where \( s(z) = z^2 \), then it is not hard to see that \( f = F \circ h \). Since \( F \) is not one-to-one, by Schwarz’s lemma, \( |F'(0)| < 1 \) (if \( |F'(0)| = 1 \), then \( F(z) = \lambda z \), which is one-to-one). Thus, \( M = |f'(a)| = |F'(h(a))||h'(a)| = |F'(0)||h'(a)| < |h'(a)| \). So \( |h'(a)| > M \), which contradicts that \( h \in F \). Consequently, \( f \) has to be surjective. \[\square\]