Signed Measure: Hahn and Jordan decomposition

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Recall that an abstract measure is a nonnegative extended real valued function defined on a measurable space \((X, \mathcal{B})\). In contrast, a signed measure may also take negative values. A signed measure \(\nu\) by definition satisfies the following properties:

1. \(\nu\) assumes at most one of the values \(-\infty\) and \(\infty\).
2. \(\nu(\emptyset) = 0\).
3. \(\nu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \nu(E_i)\) for any sequence \(\langle E_i \rangle\) of disjoint measurable sets. And we require that if \(\nu(\bigcup_{i=1}^{\infty} E_i)\) is finite then the series converges absolutely.

Property (1) suffices to avoid \(-\infty - \infty\). Property (2) makes sure we have at least one positive set (this notion will be explained shortly). Property (3) is crucial for the existence of a positive set with positive measure contained in a measurable set with finite positive measure.

We say a set \(A\) is positive w.r.t. a signed measure \(\nu\) if \(A\) is measurable and all measurable subsets of \(A\) have nonnegative measure. Similarly, \(B\) is called negative if it is measurable and every measurable subset of it has nonpositive measure. A set \(C\) is called a null set if it is both positive and negative. Notice that positivity, negativity and nullity are notions involving not just the single set itself but a spectrum of its subsets.

Obviously, every measurable subset of a positive set is positive and a union of a countable collection of disjoint positive sets is also positive (by property (3)). If we have a countable collection of positive sets \(\langle P_i \rangle\) which are not necessarily disjoint, to show their union is still positive, let \(E\) be an arbitrary measurable set of \(\bigcup P_i\), and define \(E_i = E \cup P_i \setminus \bigcup_{n=1}^{i-1} P_n\). Then \(\langle E_i \rangle\) is a sequence of disjoint measurable sets whose union is \(E\) and \(\nu\) is measurable and all measurable subsets of \(E\) are contained in positive \(P_i\). Thus, \(\nu(E) = \nu(\bigcup E_i) = \sum \nu(E_i) \geq 0\) by (3).

It will be eventually proved that given a signed measure space \((X, \mathcal{B}, \nu)\), there is a positive set \(A\) and a negative set \(B\) such that \(A\) and \(B\) partition \(X\). This is called the Hahn Decomposition Theorem. Before proving that, we justify an important lemma.

**Lemma 1.** Let \(E\) be a measurable set having finite positive measure. Then there is a positive set \(P\) having positive measure contained in \(E\).

**Proof.** If \(E\) itself is positive, then we are done. Suppose \(E\) is not positive, then it has a measurable subset of negative measure. Let \(n_1 \in \mathbb{Z}^+\) be the smallest such that there is a measurable set \(E_1\) with \(\nu(E_1) < -\frac{1}{n}\) (If we make \(n\) smaller, there is no \(E_1\) satisfying the condition.). Then consider \(E \setminus E_1\). First, \(\nu(E \setminus E_1) = \nu(E) - \nu(E_1) > \nu(E) > 0\). If \(E \setminus E_1\) is positive, then we turn off the proof. Inductively, if \(E \setminus \bigcup_{i=1}^{k-1} E_i\) is not positive, let \(n_k \in \mathbb{Z}^+\) be the smallest such that there is a measurable set \(E_k\) with \(\nu(E_k) < -\frac{1}{n_k}\). If we never stop, let \(P = E \setminus \bigcup_{i=1}^{\infty} E_i\). Then \(E = P \cup \bigcup E_i\). Notice that \(P\) and \(\langle E_i \rangle\) are disjoint, so by (3) \(\nu(E) = \nu(P) + \sum \nu(E_i)\). But \(\nu(E) < \infty\), by definition (3), \(\sum \nu(E_i)\) is absolutely convergent. Hence, \(\lim_{n \to \infty} n_i = 0\). Let \(\epsilon > 0\). So there is \(k\) such that \(\frac{1}{n_k-1} < \epsilon\). Hence, \(-\epsilon < -\frac{1}{n_k-1}\). Notice that \(P\) is contained in \(E \setminus \bigcup_{i=1}^{k-1} E_i\). If \(A\) contains a measurable set which is henceforth contained in \(E \setminus \bigcup_{i=1}^{k-1} E_i\) with measure less than \(-\epsilon\) and thus less than \(-\frac{1}{n_k-1}\) (and then less than \(-\frac{1}{n_k}\); this is useless), then \(n_k\) is no longer the smallest positive integer making the existence of a measurable set having measure less than the negative of its reciprocal (but \(n_k - 1\) is!).
Therefore, $P$ contains no measurable set having measure less than $-\epsilon$. Since $\epsilon$ is arbitrary, $P$ contains no measurable set having negative measure. This shows $P$ is positive.

We also know $\nu(P) = \nu(E) - \sum \nu(E_i) > \nu(E) > 0$. This completes the proof. \qed

We don’t need to consider the case when $n_k = 1$ so that there is no way to make $n_k - 1 \in \mathbb{Z}_+$ since we only concern the case when $\epsilon$ is very small.

Now we prove the Hahn Decomposition Theorem.

**Theorem 2** (Hahn Decomposition Theorem). Let $(X, \mathcal{B}, \nu)$ be a signed measure space. Then there is a positive set $P$ and a negative set $N$ such that $X = P \cup N$ and $P \cap N = \emptyset$.

**Proof.** We may assume $\nu$ never takes $\infty$. Let $p = \sup_{P^+} \nu$ (positive) positive, $p \geq 0$. By the definition of sup, there is a sequence of positive sets $(P_i)$ such that $p = \lim_{i \to \infty} \nu(P_i)$. Let $P = \bigcup P_i$. So $P$ is positive. Hence, $p \geq \nu(P)$. But for any $i$, $\nu(A_i) + \nu(A \setminus A_i) \geq \nu(A_i)$. Therefore, $\nu(A) \geq p$. Thus, $0 \leq \nu(A) = p < \infty$. Let $N = X \setminus P$. We claim that $N$ is negative and then we are done. Suppose $E$ is a positive subset of $N$. So $E$ and $P$ are disjoint and $E \cup P$ is positive. Hence, $p \geq \nu(E \cup P) = \nu(E) + \nu(p) = \nu(E) + p$. this shows $\nu(E) = 0$. Therefore, $N$ contains no positive subsets of positive measure. Consequently, by the contrapositive of the lemma, $N$ contains no subsets of positive measure. \qed

The Hahn decomposition of a measurable space associated with a signed measure exists by the above theorem. However, it is not unique. For example. Let $m$ be the Lebesgue measure on $\mathbb{R}$. Define $\nu_1(E) = m(E \cap [-1,0])$ and $\nu_2(E) = m(E \cap [0,1])$. Let $\nu = \nu_1 - \nu_2$. First of all, $(-\infty,0)$ and $(0,\infty)$ is a Hahn decomposition because for any measurable $E \subset (-\infty,0)$, $\nu(E) = \nu_1(E \cap [-1,0]) - \nu_2(E \cap [0,1]) = \nu_1(E \cap [-1,0]) \geq 0$ and any $F \subset (0,\infty)$, $\nu(F) = \nu_1(\emptyset) - \nu_2(F \cap [0,1]) \leq 0$. On the other hand, it can be similarly checked that $[-1,0]$ and $(-\infty,-1) \cup (0,\infty)$ is another Hahn decomposition. Nonetheless, Hahn decomposition is unique except for null sets. In the above example $(-\infty,-1)$ is a null set and the difference between the two composition is exactly due to the freedom of $(-\infty,-1)$

If $\nu$ is a signed measure and $P$ and $N$ is a Hahn decomposition. Then we may define $\nu^+ \nu$ by $\nu^+(E) = \nu(E \cap P)$ and $\nu^-$ by $\nu^-(E) = -\nu(E \cap N)$.So $\nu = \nu^+ - \nu^-$. Notice that $\nu^+$ and $\nu^-$ are measures and they are mutually singular in the sense that there is binary partition $\{A, B\}$ of $X$ (in this case $A = N$ and $B = P$) such that $\nu^+(A) = \nu^-(B) = 0$. A decomposition of the measure $\nu$ as a difference of two mutually singular measures $\nu^+$ and $\nu^-$ exemplified previously is called a **Jordan decomposition**. However, it is worthy to note that although we defined $\nu^+$ and $\nu^-$ in terms of a particular Hahn decomposition of $X$, the Jordan decomposition is actually independent of the Hahn decomposition and there is only one pair of such decomposition. This may seem a little confusing. To see it clearly (the Jordan decomposition is unique), let $\nu = \nu^+ - \nu^-$ be a Jordan decomposition. Then by definition, there is a partition $A$ and $B$ of $X$ such that $\nu^+(A) = \nu^-(B) = 0$. We claim $\{B,A\}$ is a Hahn decomposition. [Let $E \subset B$, then $\nu^-(E) \leq \nu^-(B) = 0$, i.e., $\nu^-(E) = 0$. Thus, $\nu(E) = \nu^+(E) - \nu^-(E) \geq 0$. This shows $B$ is positive. Similarly, $A$ is negative.] Then since Hahn decomposition is up to null sets, the Jordan decomposition is unique.