New bounds for Betti numbers of semi-algebraic sets and algorithms for computing them

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- Subsets of $\mathbb{R}^k$ defined by a formula involving a finite number of polynomial equalities and inequalities.
- A basic semi-algebraic set is one defined by a conjunction of weak inequalities of the form $P \geq 0$.
- They arise as configurations spaces (in robotic motion planning, molecular chemistry etc.), CAD models and many other applications in computational geometry.

Part I

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Bounds on the Complexity of Semi-algebraic Sets

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Uniform bounds on the number of connected components, Betti numbers etc.

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Dimension of the ambient space : $\mathbf{k}$

Dimension of the set itself : $\mathbf{k}'$

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- $\beta_i(S)$ is the rank of the $H^i(S)$ (the i-th co-homology group of S).
- $\beta_0(S) =$ the number of connected components.
- $\beta_i(S) =$ the number of i-cycles that do not bound.

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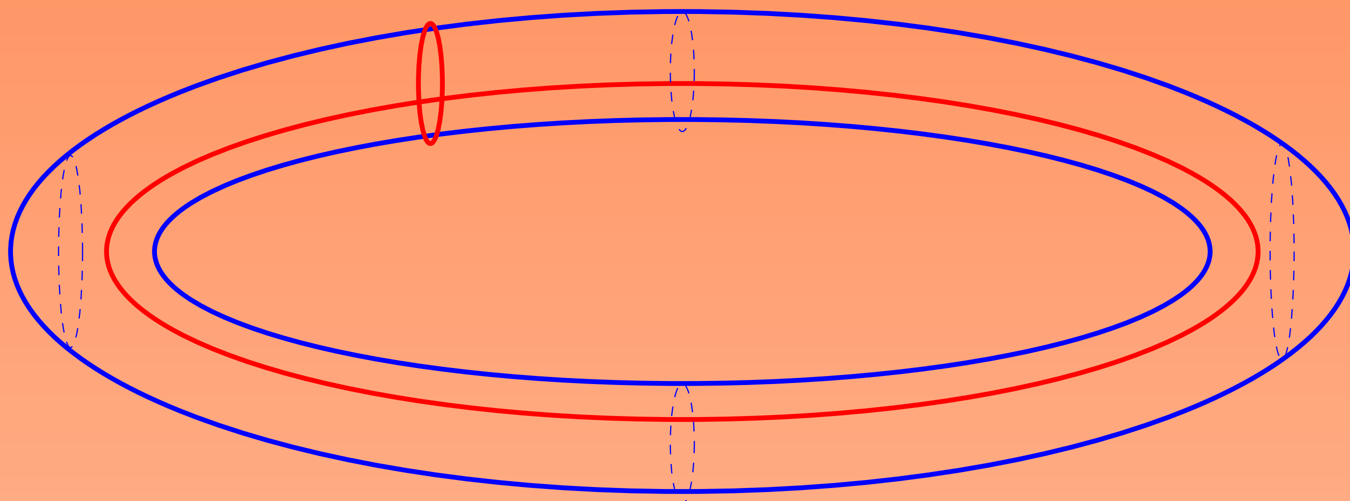

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- $\beta_i(T) = 0, i > 2.$

Classical Result on the Topology of Semi-algebraic Sets

Theorem 1. *(Oleinik and Petrovsky, Thom, Milnor) Let* $S \subset \mathbb{R}^k$ *be the set defined by the conjunction of* n *inequalities,*

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in \mathbb{R}[X_1, \ldots, X_k],$$

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$$\sum_{\mathbf{i}} \beta_{\mathbf{i}}(\mathbf{S}) = \mathbf{nd(2nd-1)^{k-1}} = \mathbf{O(nd)^k}.$$

Tightness

The above bound is actually quite tight. Example: Let

$$P_i = L_{i,1}^2 \cdots L_{i,\lfloor d/2 \rfloor}^2 - \epsilon,$$

where the L_{ij}'s are generic linear polynomials and $\epsilon > 0$ and sufficiently small. The set S defined by $P_1 \geq 0, \ldots, P_n \geq 0$ has $\Omega(nd)^k$ connected components and hence $\beta_0(S) = \Omega(nd)^k$.

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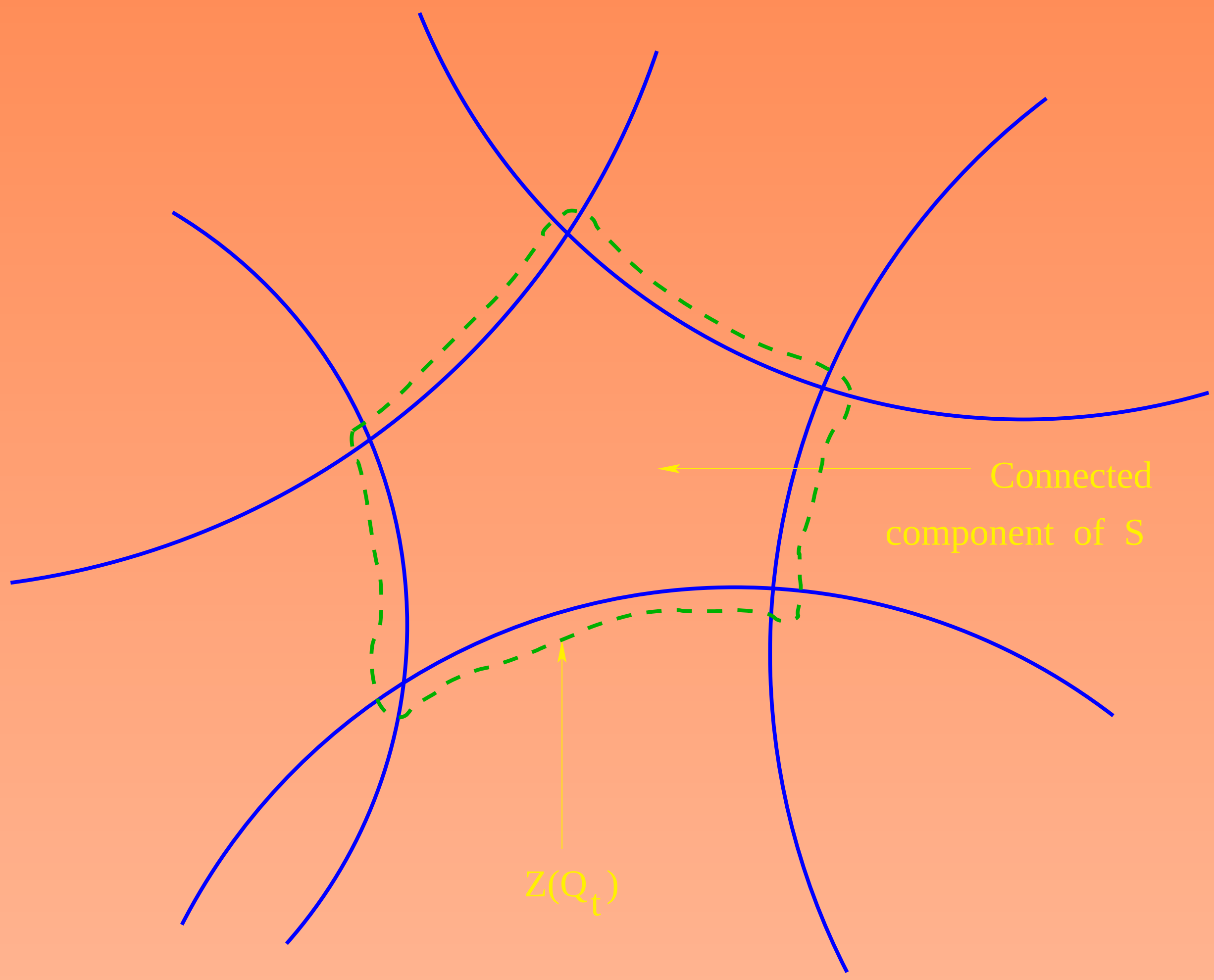
- Cannot construct examples such that $\beta_i(S) = \Omega(nd)^k$ for $i > 0$.

- The technique used for proving the above result does not help:

  Replace the semi-algebraic set S by another set bounded by a smooth algebraic hypersurface of degree $2nd$ having the same homotopy type as S. Then bound the Betti numbers of this hypersurface using Morse theory and the Bezout bound on the number of solutions of a system of polynomial equations.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Graded Bounds

Theorem 2. *(B, 2001) Let $S \subset \mathbb{R}^k$ be the set defined by the conjunction of n inequalities,*

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$$\beta_{\mathbf{i}}(\mathbf{S}) \leq \binom{\mathbf{n}}{\mathbf{k'-i}} (\mathbf{4d})^{\mathbf{k}}.$$

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Theorem 3. *(B, 2001) Let* $S \subset \mathbb{R}^k$ *be the set defined by the disjunction of* n *inequalities,*

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Example: $\mathbf{X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0}$.

Betti Numbers of Sign Patterns I

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- Let $b_i(\sigma)$ denote the i-th Betti number of the realization of σ, and let $b_i(\mathcal{Q}, \mathcal{P}) = \sum_\sigma b_i(\sigma)$.

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- Let $b_i(d, k, k', n)$ be the maximum of $b_i(\mathcal{Q}, \mathcal{P})$ over all $\mathcal{Q}, \mathcal{P}$ where $\mathcal{Q}$ and $\mathcal{P}$ are finite subsets of of $\mathbb{R}[X_1, \ldots, X_k]$, whose elements have degree at most d, $\#(\mathcal{P}) = n$ and the algebraic set $Z(\mathcal{Q})$ has dimension k'.

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- Previously known (B, Pollack, Roy (1995))

$$\mathbf{b_0(d,k,k',n)} = \binom{\mathbf{4n}}{\mathbf{k'}}\mathbf{d(2d-1)^{k-1}} = \binom{\mathbf{n}}{\mathbf{k'}}\mathbf{O(d)^k}.$$

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Theorem 5. *(B, Pollack,Roy, 2002)*

$$\mathbf{b_i(d,k,k',n) \leq \sum_{0\leq j\leq k'-i} \binom{n}{j} 4^j d(2d-1)^{k-1} = \binom{n}{k'-i} O(d)^k.}$$

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Applications ?

Proofs

Uses spectral sequences associated to the Mayer-Vietoris double complex.

Generalized Mayer-Vietoris Exact Sequence

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- Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex A such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the $\mathbb{R}$-vector space of i co-chains of A, and $C^*(A) = \oplus_i C^i(A)$.

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- We will denote by $A_{\alpha_0,\ldots,\alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.
- The following sequence of homomorphisms is exact.

$$0 \longrightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0<\alpha_1} C^*(A_{\alpha_0,\alpha_1})$$

$$\cdots \xrightarrow{\delta} \prod_{\alpha_0<\cdots<\alpha_p} C^*(A_{\alpha_0,\ldots,\alpha_p}) \cdots \xrightarrow{\delta} \prod_{\alpha_0<\cdots<\alpha_{p+1}} C^*(A_{\alpha_0,\ldots,\alpha_{p+1}}) \cdots \xrightarrow{\delta} \cdots$$

Mayer-Vietoris Double Complex I

We now consider the following bigraded double complex $\mathcal{M}^{p,q}$, with a total differential $D = \delta + (-1)^p d$, where

$$\mathcal{M}^{p,q} = \prod_{\alpha_0,\ldots,\alpha_p} C^q(A_{\alpha_0,\ldots,\alpha_p}).$$

and ...

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{\alpha_0} C^3(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^3(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^3(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{\alpha_0} C^2(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^2(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^2(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \\
 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
0 & \longrightarrow & \prod_{\alpha_0} C^1(A_{\alpha_0}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1} C^1(A_{\alpha_0,\alpha_1}) & \xrightarrow{\delta} & \prod_{\alpha_0<\alpha_1<\alpha_2} C^1(A_{\alpha_0,\alpha_1,\alpha_2}) & \longrightarrow & \\
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 & & \uparrow d & & \uparrow d & & \uparrow d & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Double Complex

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
C^{0,2} & \xrightarrow{\delta} & C^{1,2} & \xrightarrow{\delta} & C^{2,2} & \xrightarrow{\delta} & \cdots \\
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$$

The Associated Total Complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
\diagdown & & \uparrow d & \diagdown & \uparrow d & \diagdown & \uparrow d & \diagdown & \\
 & \xrightarrow{\delta} & C^{p-1,q+1} & \xrightarrow{\delta} & C^{p,q+1} & \xrightarrow{\delta} & C^{p+1,q+1} & \xrightarrow{\delta} & \cdots \\
\diagdown & & \uparrow d & \diagdown & \uparrow d & \diagdown & \uparrow d & \diagdown & \\
 & \xrightarrow{\delta} & C^{p-1,q} & \xrightarrow{\delta} & C^{p,q} & \xrightarrow{\delta} & C^{p+1,q} & \xrightarrow{\delta} & \cdots \\
\diagdown & & \uparrow d & \diagdown & \uparrow d & \diagdown & \uparrow d & \diagdown & \\
 & \xrightarrow{\delta} & C^{p-1,q-1} & \xrightarrow{\delta} & C^{p,q-1} & \xrightarrow{\delta} & C^{p+1,q-1} & \xrightarrow{\delta} & \cdots \\
\diagdown & & \uparrow d & \diagdown & \uparrow d & \diagdown & \uparrow d & \diagdown & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

$$D = d \pm \delta$$

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- $E_{r+1} = H(E_r, d_r)$,
- $E_\infty = H^*(\text{Associated Total Complex})$.

Pictorially ...

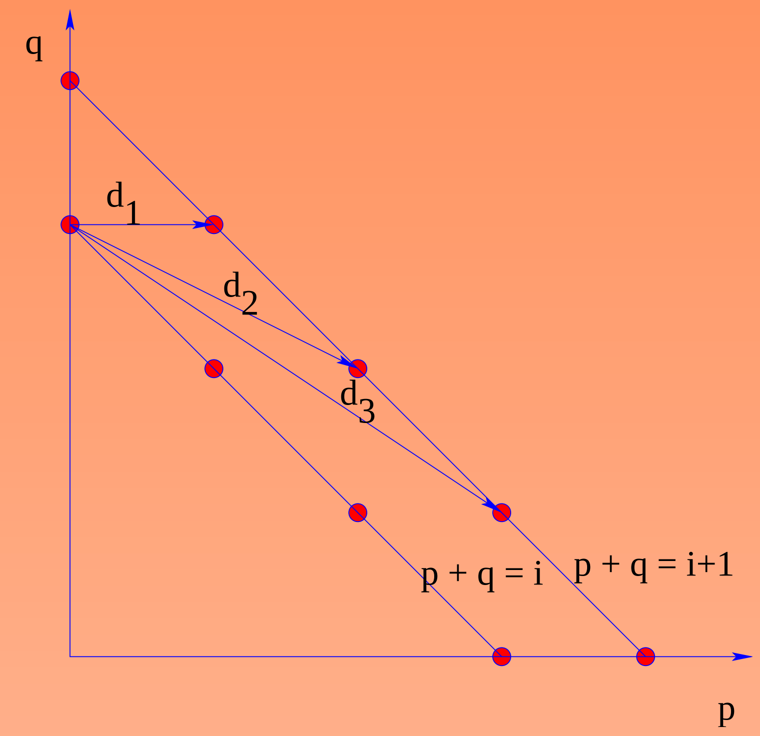

Figure 1: The differentials d_r in the spectral sequence (E_r, d_r)

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- $$\mathbf{E'_1} = \mathbf{H_d}(\mathcal{M}), \;\; \mathbf{E'_2} = \mathbf{H}_\delta\mathbf{H_d}(\mathcal{M})$$

$$E_1 = \begin{array}{ccc} \vdots & \vdots & \vdots \\ C^3(A) & 0 & 0 \\ C^2(A) & 0 & 0 \\ C^1(A) & 0 & 0 \\ C^0(A) & 0 & 0 \end{array}$$

$$E_2 = \begin{array}{ccc} \vdots & \vdots & \vdots \\ H^3(A) & 0 & 0 \\ H^2(A) & 0 & 0 \\ H^1(A) & 0 & 0 \\ H^0(A) & 0 & 0 \end{array}$$

The degeneration of this sequence at E_2 shows that

$$\mathbf{H}^*_{\mathbf{D}}(\mathcal{M}) \cong \mathbf{H}^*(\mathbf{A}).$$

$$E_1' = \begin{matrix} \vdots & \vdots & \vdots \\ \prod_{\alpha_0} H^3(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^3(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^3(A_{\alpha_0, \\ \prod_{\alpha_0} H^2(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^2(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^2(A_{\alpha_0, \\ \prod_{\alpha_0} H^1(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^1(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^1(A_{\alpha_0, \\ \prod_{\alpha_0} H^0(A_{\alpha_0}) & \prod_{\alpha_0<\alpha_1} H^0(A_{\alpha_0,\alpha_1}) & \prod_{\alpha_0<\alpha_1<\alpha_2} H^0(A_{\alpha_0, \end{matrix}$$

Lemma 1

Lemma 6. *Let A be a finite simplicial complex and $A_1, \ldots, A_n$ subcomplexes of A such that $A = A_1 \cup \cdots \cup A_n$. Suppose that for every ℓ, $0 \leq \ell \leq i$, and for every $(\ell + 1)$ tuple $A_{\alpha_0}, \ldots, A_{\alpha_\ell}$, $\beta_{i-\ell}(A_{\alpha_0,\ldots,\alpha_\ell}) \leq M$. Then, $\beta_i(A) \leq \sum_{0 \leq \ell \leq i} \binom{n}{\ell+1} M$.*

Lemma 2

Lemma 7. *Let* $P_1, \ldots, P_l \in R[X_1, \ldots, X_k]$, $deg(P_i) \leq d$, *and* $l \leq k$. *Let* S *be the set defined by the conjunction of the inequalities* $P_i \geq 0$. *Assume that* S *is bounded. Then,* $\sum_i \beta_i(S) = (4d)^k$.

Theorem 3 follows.
Theorem 2 follows by a dual argument.
Theorem 4 follows using a result of Barvinok (1995).

Part II

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Algorithms in Computational Geometry.

Arrangements in Computational Geometry

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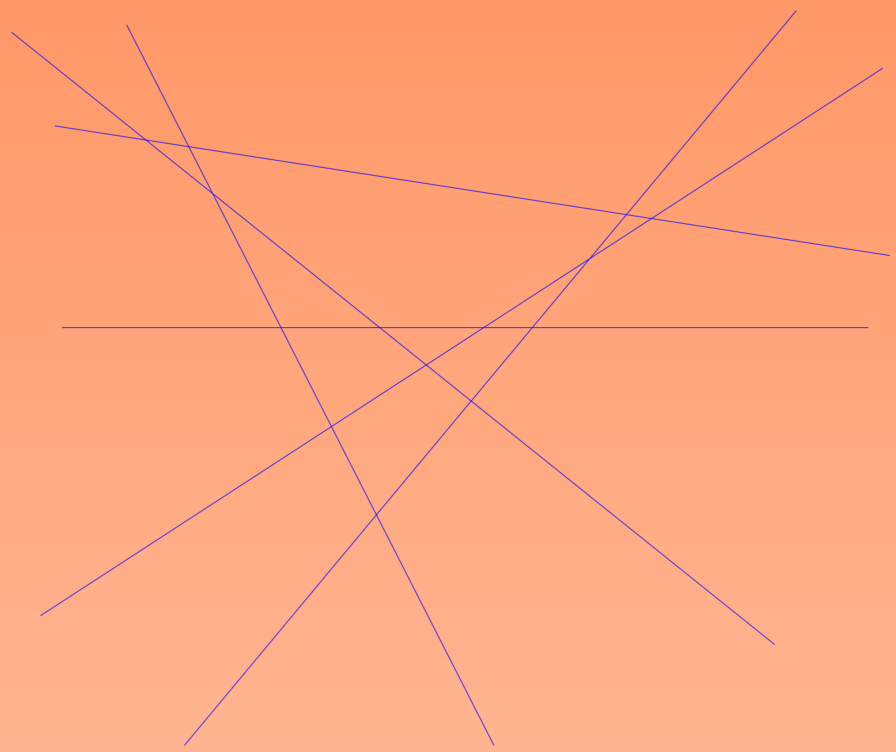
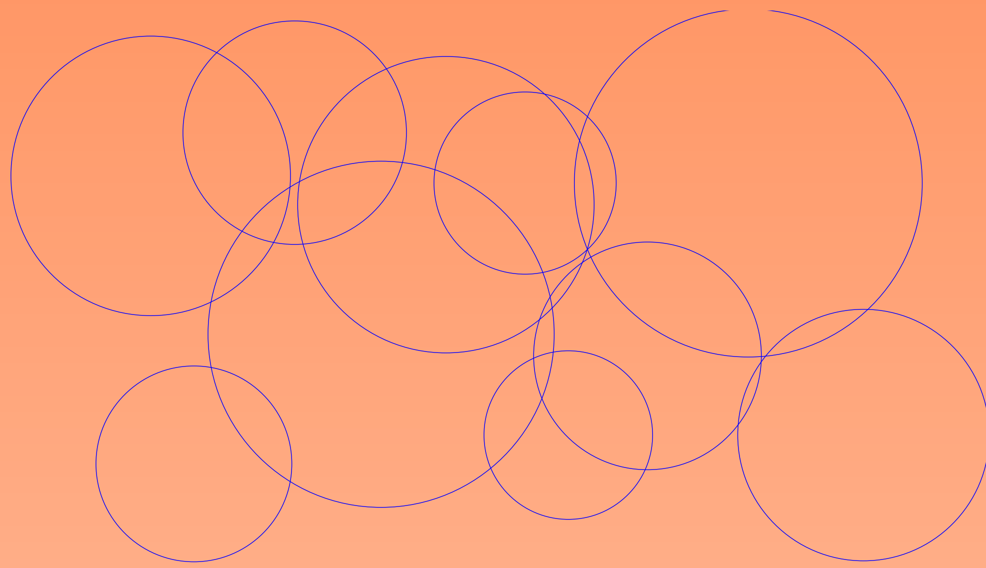
- Arrangements of lines in the plane, or more generally hyperplanes in $\mathbb{R}^k$.
- Arrangements of balls or simplices in $\mathbb{R}^k$.
- Arrangements of semi-algebraic objects in $\mathbb{R}^k$, each defined by a fixed number of polynomials of constant degree.

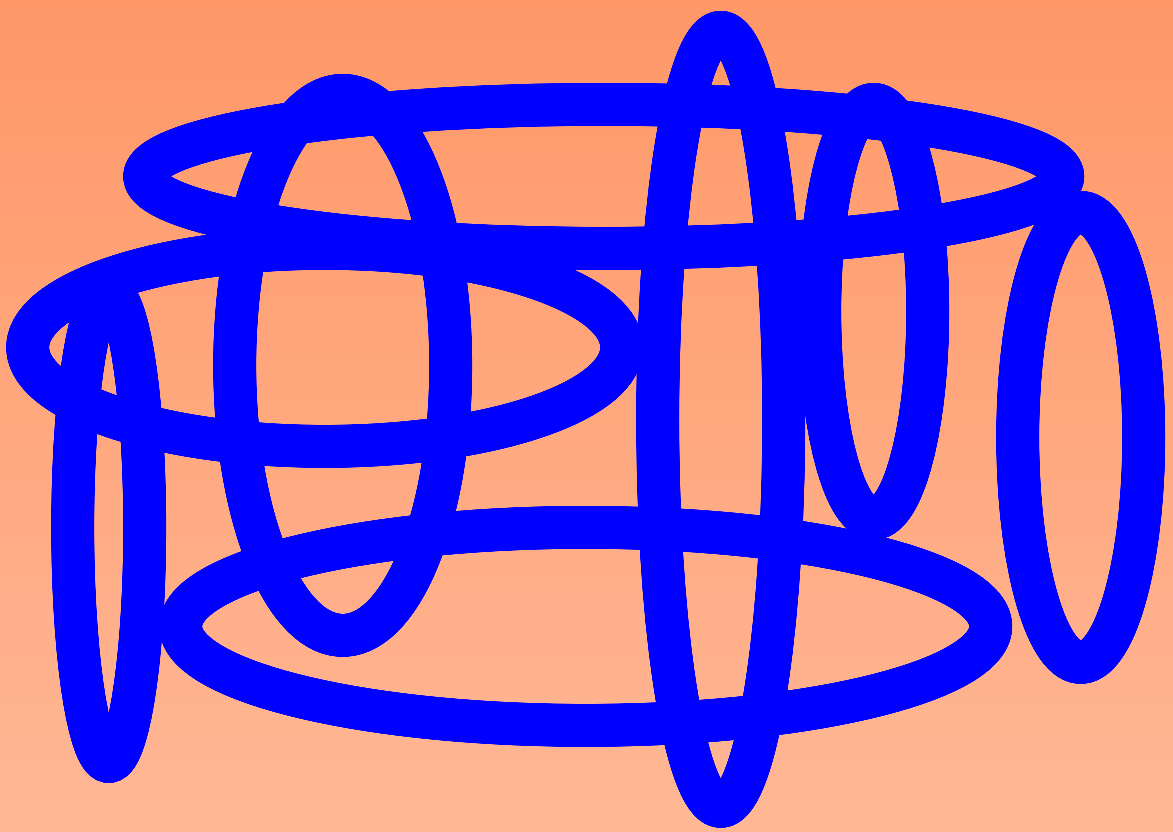
Arrangements of lines in the $\mathbb{R}^2$



Arrangement of circles in $\mathbb{R}^2$



Arrangement of tori in $\mathbb{R}^3$



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- Computing the Betti numbers of triangulated manifolds (Edelsbrunner, Dey, Guha et al).

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- We only count the number of algebraic operations and ignore the cost of doing linear algebra.

Two Approaches

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Global

vs

Local

First Approach (Global): Using Triangulations

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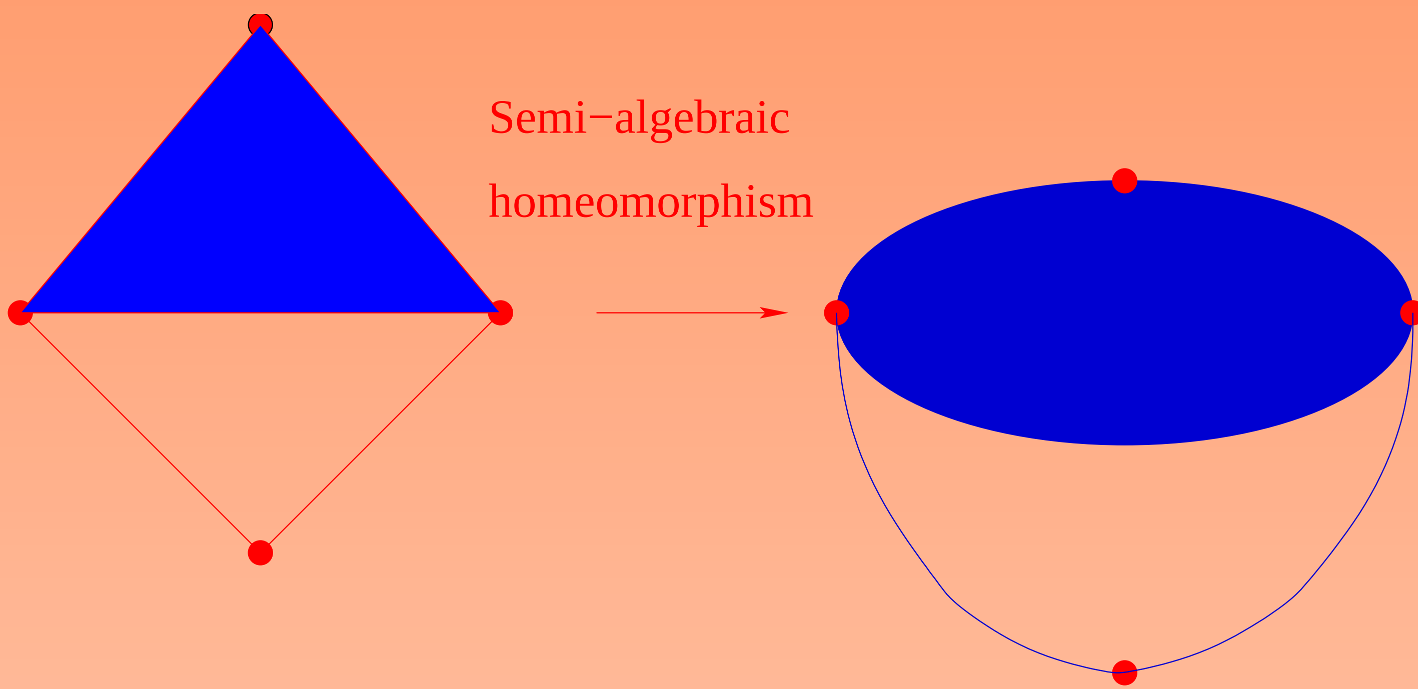
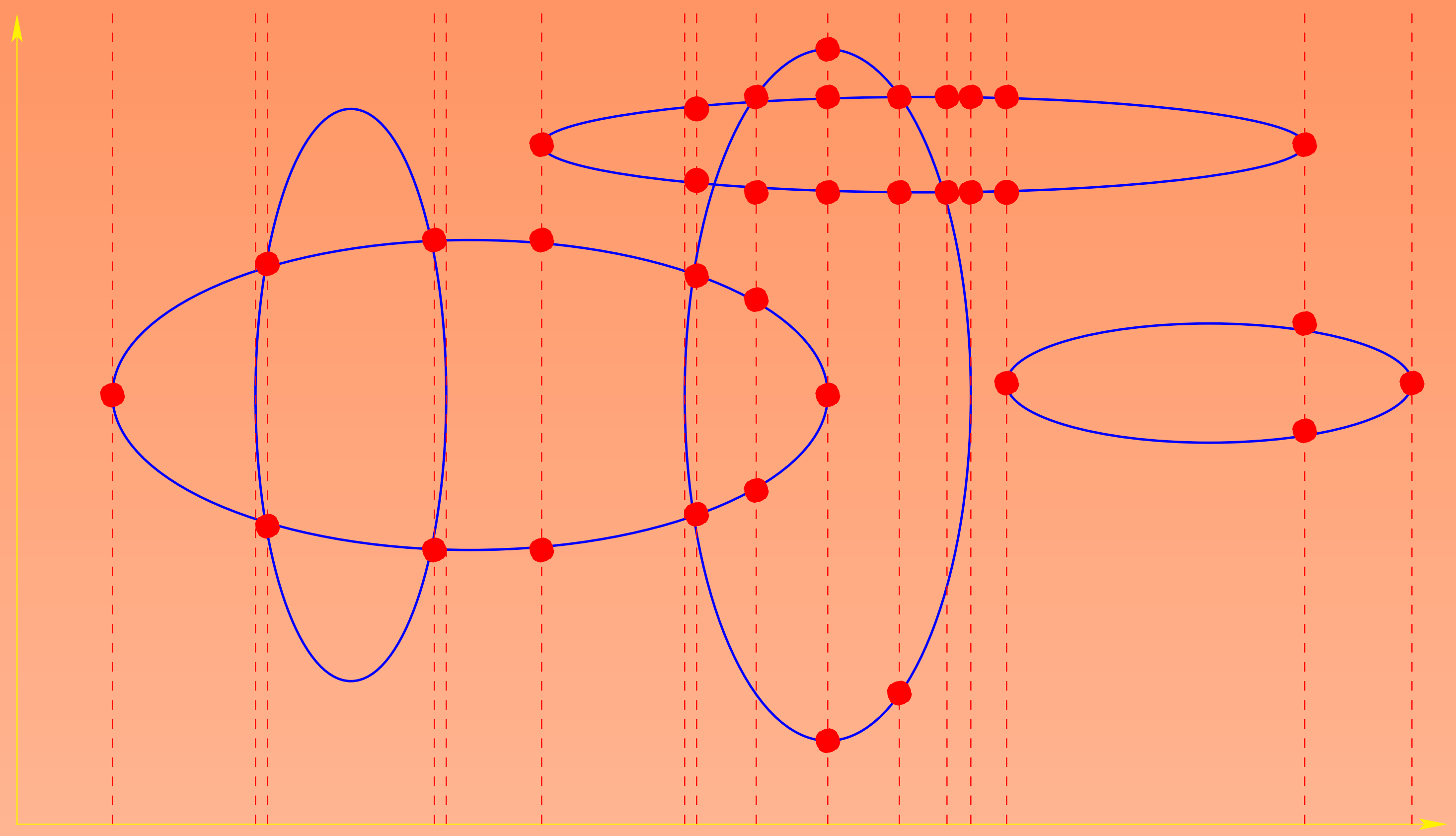
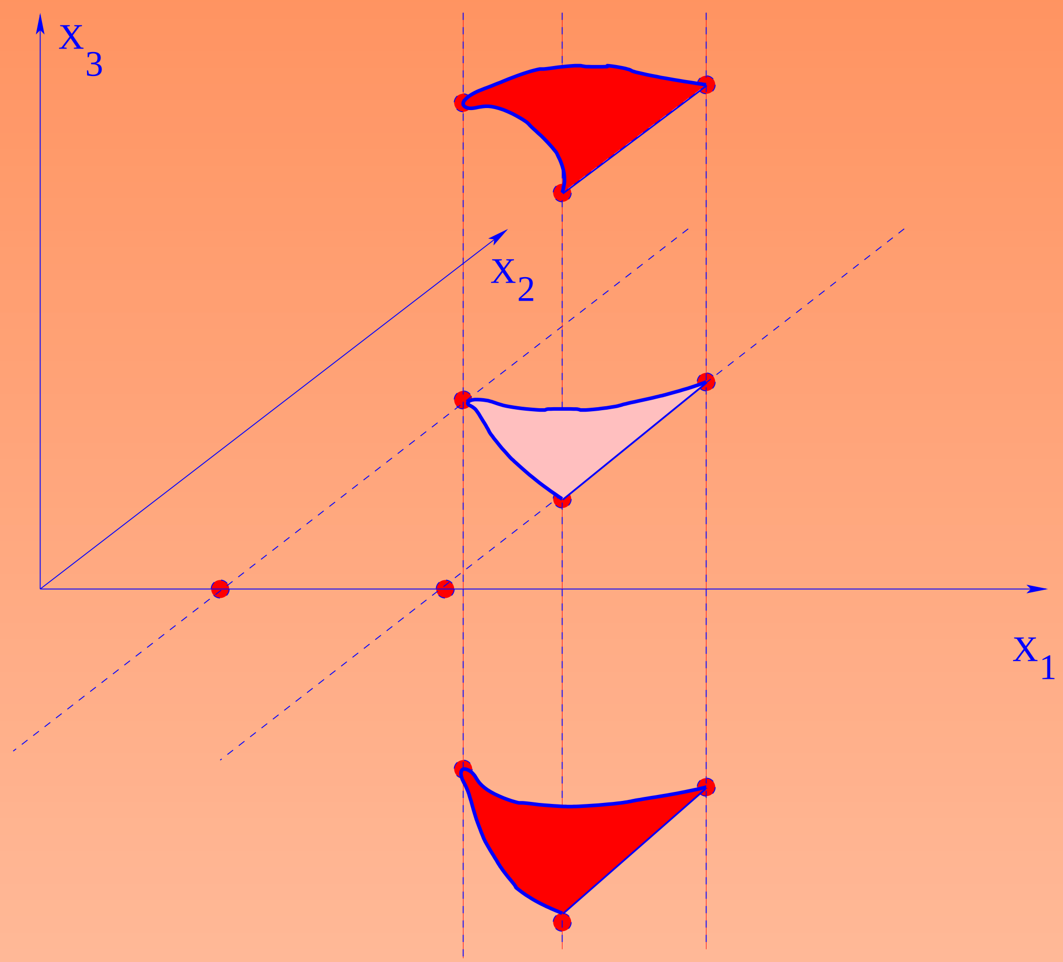


Using Collin's Cylindrical Algebraic Decomposition



Picture of a cylinder



Computing Betti Numbers using Global Triangulations

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- First triangulate the arrangement using *Cylindrical algebraic decomposition* and then compute the Betti numbers of the corresponding simplicial complex.
- But ... CAD produces $O(n^{2^k})$ simplices in the worst case.

Second Approach (Local): Using the Nerve Complex

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- The homology groups of the union are then isomorphic to the homology groups of a combinatorially defined complex called the *nerve complex*.

The Nerve Complex

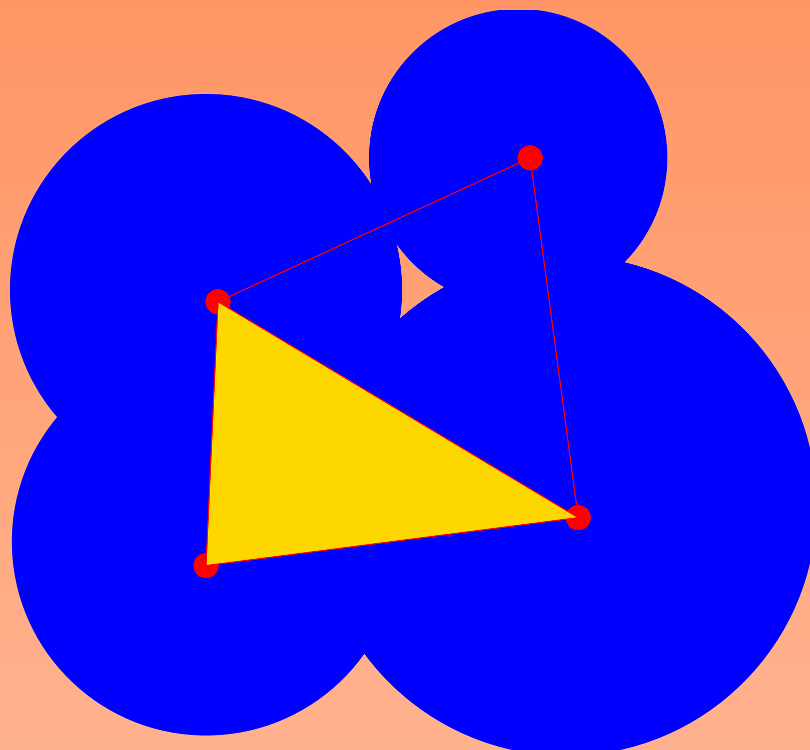

Figure 2: The nerve complex of a union of disks

Computing the Betti Numbers via the Nerve Complex (local algorithm)

- The nerve complex has n vertices, one vertex for each set in the union, and a simplex for each *non-empty* intersection among the sets.

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- The nerve complex has n vertices, one vertex for each set in the union, and a simplex for each *non-empty* intersection among the sets.

- Thus, the $(\ell+1)$-skeleton of the nerve complex can be computed by testing for non-emptiness of each of the possible $\sum_{1 \leq j \leq \ell+2} \binom{n}{j} = O(n^{\ell+2})$ at most $(\ell+2)$-ary intersections among the n given sets.

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- If the sets are such that the topology of the "small" intersections are controlled, then
- we can use the *Leray spectral sequence* as a substitute for the nerve lemma.
- The algorithmic version gives the first efficient algorithm for computing the Betti numbers, without the double-exponential complexity entailed in CAD.

Main Result

Theorem 8. *Let $S_1, \ldots, S_n \subset \mathbb{R}^k$ be compact semi-algebraic sets of constant description complexity and let $S = \cup_{1 \leq i \leq n} S_i$, and $0 \leq \ell \leq k-1$. Then, there is an algorithm to compute $\beta_0(S), \ldots, \beta_\ell(S)$, whose complexity is $O(n^{\ell+2})$.*

The Algorithm

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- In order to compute β_ℓ, we only need to compute upto $E'_{\ell+2}$.*But the punchline is that:*

- In order to compute the differentials $d_r, 1 \leq r \leq \ell+1$, it suffices to have *independent triangulations of the different unions taken upto $\ell+2$ at a time.*

- For instance, it should be intuitively clear that in order to compute $\beta_0(\cup_i S_i)$ it suffices to triangulate pairs.

Spectral Sequences

Where do they come from ?

Filtration I

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d & \searrow & \uparrow d & \searrow & \uparrow d & \searrow & \\
C^{k,q+1} & \xrightarrow{\delta} & C^{k+1,q+1} & \xrightarrow{\delta} & C^{k+2,q+1} & \xrightarrow{\delta} & \cdots \\
\uparrow d & \searrow & \uparrow d & \searrow & \uparrow d & \searrow & \\
C^{k,q} & \xrightarrow{\delta} & C^{k+1,q} & \xrightarrow{\delta} & C^{k+2,q} & \xrightarrow{\delta} & \cdots \\
\uparrow d & \searrow & \uparrow d & \searrow & \uparrow d & \searrow & \\
C^{k,q-1} & \xrightarrow{\delta} & C^{k+1,q-1} & \xrightarrow{\delta} & C^{k+2,q-1} & \xrightarrow{\delta} & \cdots \\
\uparrow d & \searrow & \uparrow d & \searrow & \uparrow d & \searrow & \\
\vdots & & \vdots & & \vdots & &
\end{array}
$$

Filtration II

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- $H_k^n = Z_k^n / Z_k^n \cap B^n$
- We thus have a decreasing filtration, $\cdots \supset H_{k-1}^n \supset H_k^n \supset H_{k+1}^n \cdots$ of the cohomology group $H^n(C, \mathrm{D})$. We denote the successive quotients H_k^n / H_{k+1}^n by $H^{k,n-k}$.

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- An element in $C^n = \oplus_{i+j=n} C^{i,j}$ will have a leading term at a position (p,q), where p denotes the smallest i such that the component at position $(i, n-i)$ does not vanish.
- Let $Z^{p,q}$ denote the set of the (p,q) components of co-cycles whose leading term is at position (p,q).

Spectral Sequence III

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$$\begin{aligned} da &= 0 \\ \delta a &= -da^{(1)} \\ \delta a^{(1)} &= -da^{(2)} \\ \delta a^{(2)} &= -da^{(3)} \\ &\vdots \end{aligned}$$

Here, $a^{(i)} \in C^{p+i,q-i}$.

Pictorially ...

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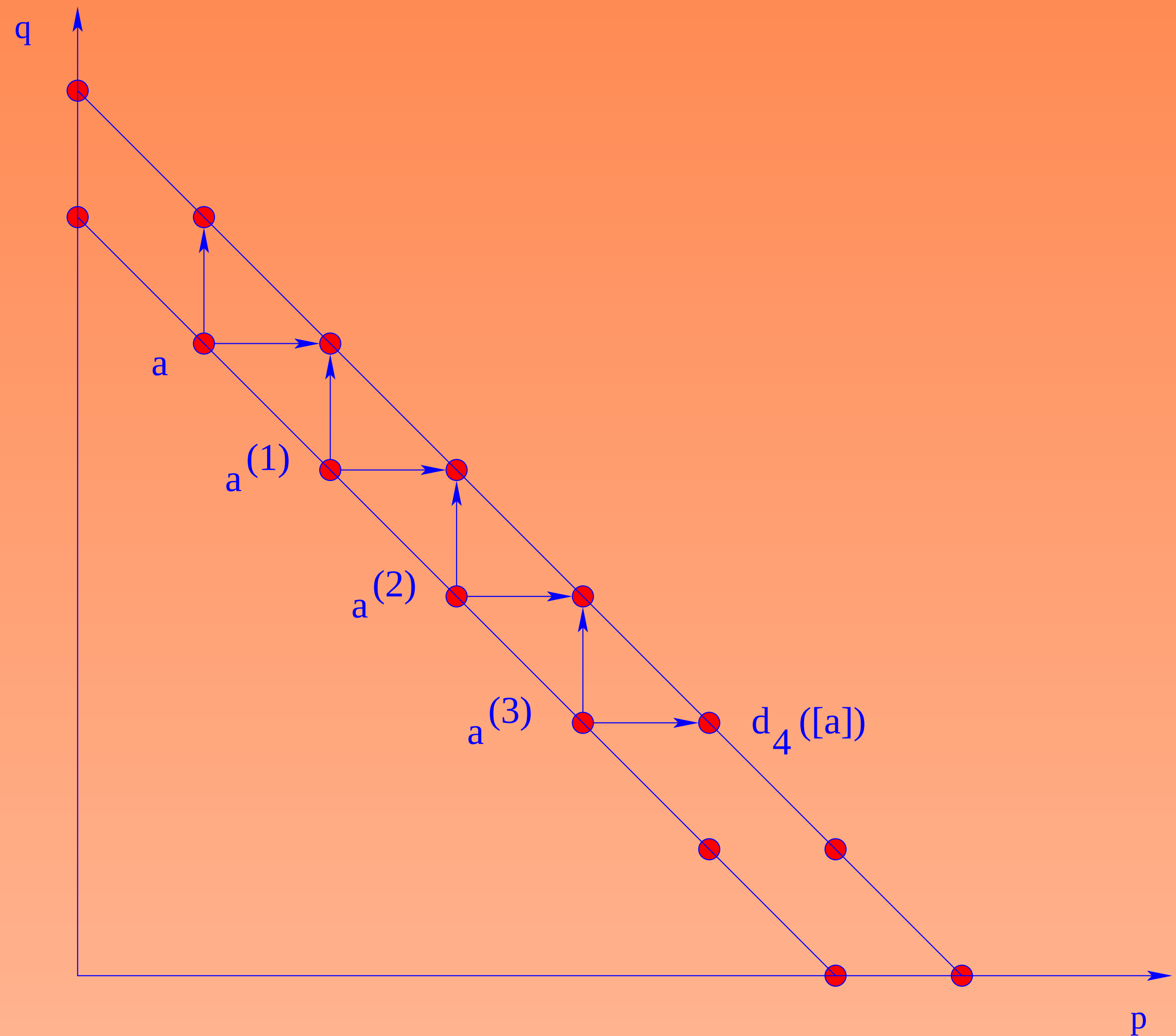

Figure 3: $\delta a^{(i)} + da^{(i+1)} = 0$

Spectral Sequence IV

Now, let $B^{p,q} \subset C^{p,q}$ consist of all b with the property that the following system of equations admits a solution.

$$\begin{gathered} db^{(0)} + \delta b^{(-1)} = b \\ db^{(-1)} + \delta b^{(-2)} = 0 \\ db^{(-2)} + \delta b^{(-3)} = 0 \\ \vdots \end{gathered}$$

Here, $b^{(-i)} \in C^{p-i,q+i-1}$.

Spectral Approximation

- Clearly, $H^{p,q} \cong Z^{p,q}/B^{p,q}$.

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- $$\mathbf{E_r^{p,q} = Z_r^{p,q} / B_r^{p,q}}.$$

- Let $[a] \in E_r^{p,q}$ for some $a \in Z_r^{p,q}$. Then, there exists $a^{(1)}, \ldots, a^{(r-1)}$ satisfying equations above. We let

$$\mathbf{d_r[a] = [\delta a^{(r-1)}] \in E_r^{p+r,q-r+1}}.$$

Pictorially ...

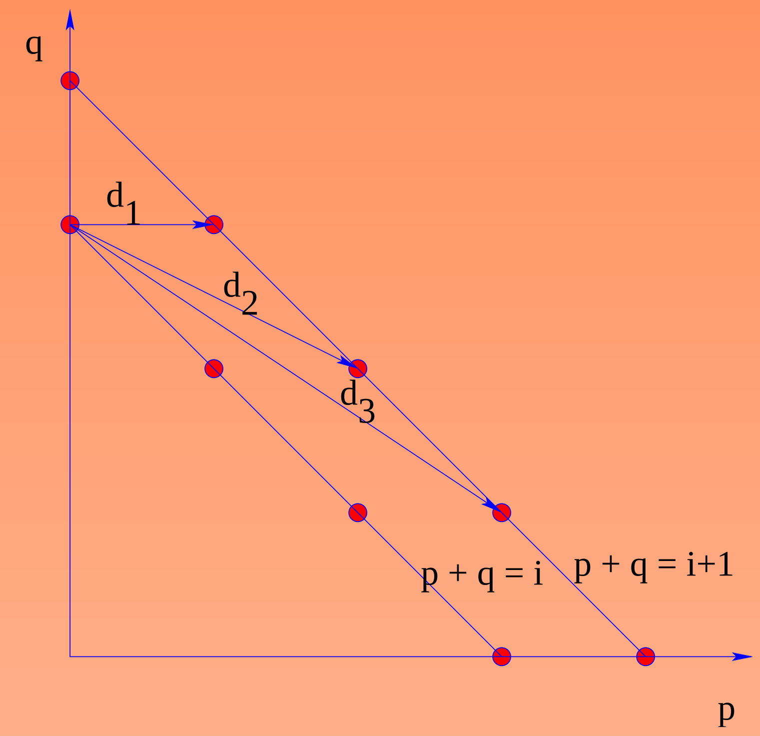

Figure 4: The differentials d_r in the spectral sequence (E_r, d_r)

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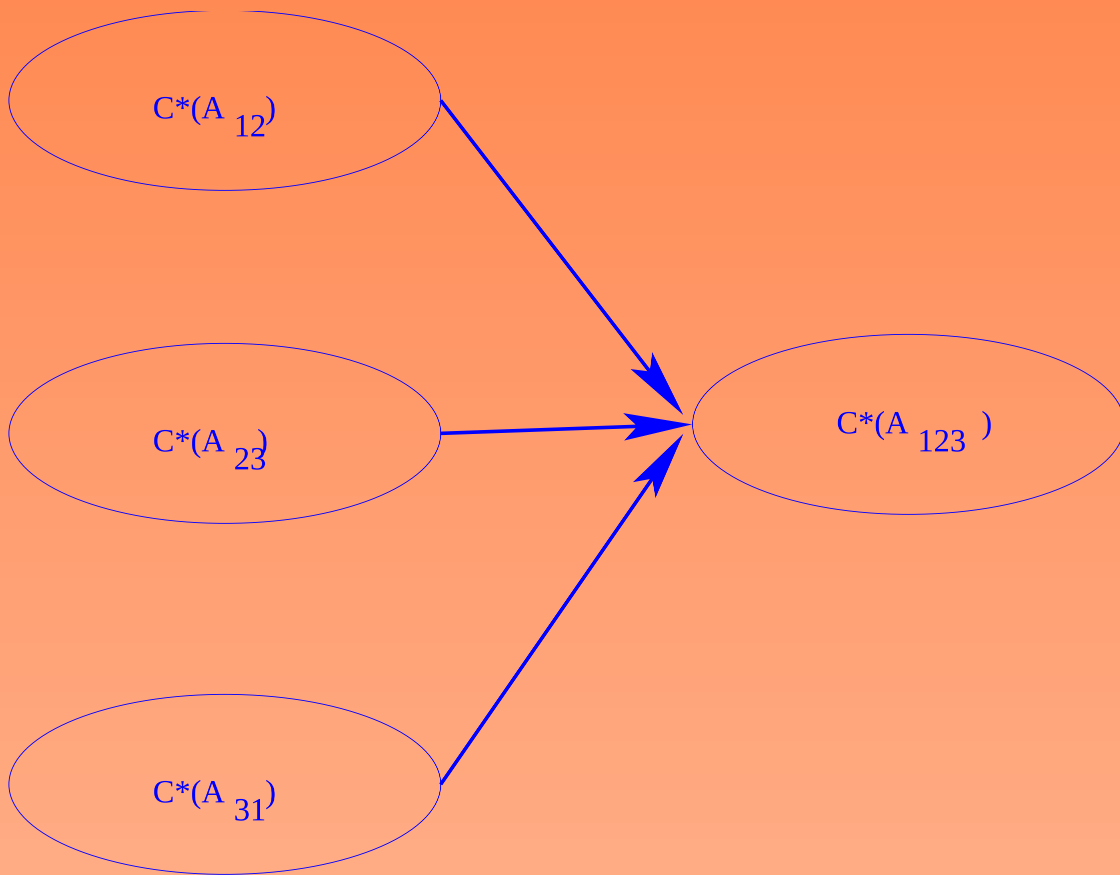
C*(A 12)
C*(A 23)
C*(A 31)
C*(A 123)

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- Same idea is applicable as a divide-and-conquer tool for computing the homology of arbitrary simplicial complexes, given a covering. What kind of efficiency do we derive ?

- To what extent does topological simplicity aid algorithms in computational geometry ?