ASSIGNMENT 5. DUE IN CLASS OCT 13, 2017.

1. (a) Let $\phi : G \to H$ be a group homomorphism, and suppose that G is abelian. Prove that $\phi(G)$ is an abelian subgroup of H. (That is the homomorphic image of an abelian group is an abelian group).
   (b) Give an example of a non-trivial (i.e. not mapping everything to the identity) of group homomorphism $\phi : G \to H$ of an abelian group G to a non-abelian group H.
   (c) Give an example of a non-trivial (i.e. not mapping everything to the identity) of group homomorphism $\phi : G \to H$ of a non-abelian group G to an abelian group H.
2. Which of the following are normal subgroups ? Justify your answer in each case.
   (a) The subgroup $4\mathbb{Z}$ of the group $\mathbb{Z}$.
   (b) The center (refer to a previous assignment) of a group G.
   (c) The subgroup $\{e, \rho\}$ of the dihedral group D_8.
   (d) The subgroup $\{e, \sigma, \sigma^2, \sigma^3\}$ of the dihedral group D_8.
   (e) The subgroup $\mathrm{SL}(2, \mathbb{R})$ of the group $\mathrm{GL}(2, \mathbb{R})$.
3. Let G be a group, and let $\mathrm{Aut}(G)$ be the set of all isomorphisms $\phi : G \to G$ (we call such isomorphisms "automorphisms of G").
   (a) Prove that $\mathrm{Aut}(G)$ is a group under the binary operation of composition.
   (b) Now consider the group Z_n (the cyclic group of order n). We say that an element $x \in Z_n$ is a *generator* of Z_n if $o(x) = n$. How many generators does Z_{10} have ?
   (c) Prove that any automorphism of Z_n must map a generator to another generator.
   (d) What is the order of the group $\mathrm{Aut}(Z_{10})$?
   (e) Identify the group $\mathrm{Aut}(Z_{10})$ (from the list of groups discussed in class).
   (f) Make a guess about the group $\mathrm{Aut}(Z_n)$ for general n ?
   (g) Justify your guess – i.e. formulate and prove a theorem.