ASSIGNMENT 5. DUE IN CLASS OCT 13, 2017.

1. (a) Let \( \phi : G \rightarrow H \) be a group homomorphism, and suppose that \( G \) is abelian. Prove that \( \phi(G) \) is an abelian subgroup of \( H \). (That is the homomorphic image of an abelian group is an abelian group).
(b) Give an example of a non-trivial (i.e. not mapping everything to the identity) of group homomorphism \( \phi : G \rightarrow H \) of an abelian group \( G \) to a non-abelian group \( H \).
(c) Give an example of a non-trivial (i.e. not mapping everything to the identity) of group homomorphism \( \phi : G \rightarrow H \) of a non-abelian group \( G \) to an abelian group \( H \).

2. Which of the following are normal subgroups ? Justify your answer in each case.
   (a) The subgroup \( 4\mathbb{Z} \) of the group \( \mathbb{Z} \).
   (b) The center (refer to a previous assignment) of a group \( G \).
   (c) The subgroup \( \{e, \rho\} \) of the dihedral group \( D_8 \).
   (d) The subgroup \( \{e, \sigma, \sigma^2, \sigma^3\} \) of the dihedral group \( D_8 \).
   (e) The subgroup \( \text{SL}(2, \mathbb{R}) \) of the group \( \text{GL}(2, \mathbb{R}) \).

3. Let \( G \) be a group, and let \( \text{Aut}(G) \) be the set of all isomorphisms \( \phi : G \rightarrow G \) (we call such isomorphisms “automorphisms of \( G \)).
   (a) Prove that \( \text{Aut}(G) \) is a group under the binary operation of composition.
   (b) Now consider the group \( Z_n \) (the cyclic group of order \( n \)). We say that an element \( x \in Z_n \) is a generator of \( Z_n \) if \( o(x) = n \). How many generators does \( Z_{10} \) have ?
   (c) Prove that any automorphism of \( Z_n \) must map a generator to another generator.
   (d) What is the order of the group \( \text{Aut}(Z_{10}) \)?
   (e) Identify the group \( \text{Aut}(Z_{10}) \) (from the list of groups discussed in class).
   (f) Make a guess about the group \( \text{Aut}(Z_n) \) for general \( n \)?
   (g) Justify your guess – i.e. formulate and prove a theorem.