PROBLEM OF THE WEEK

Solution of Problem No. 12 (Fall 2004 Series)

**Problem:** Let $a$,$b$,$c$,$d$, and $c_n$ $(n = 0, 1, 2, \ldots)$ be complex numbers such that $d \neq 0$ and

$$\frac{az+b}{z^2+cz+d} = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \ldots$$

for $|z|$ small enough.

Show that

$$\det\begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix} \Big/ \det\begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix}$$

is independent of $n$.

**Solution # 1** (by Georges Ghosh, Quebec)

From the equality $b + az = (d + cz + z^2) \cdot \displaystyle\sum_{i=0}^{+\infty} c_i z^i$

we deduce $c_0 = \dfrac{b}{d}$, $c_1 = \dfrac{ad-bc}{d^2}$ and $dc_{n+2} + cc_{n+1} + c_n = 0 \quad \forall n \geq 0$. So,

$$\begin{aligned}
c_{n+1}c_{n+3} - c_{n+2}^2 =& c_{n+1}\left(-\frac{c}{d}c_{n+2} - \frac{1}{d}c_{n+1}\right) - c_{n+2}^2 \\
=& -\frac{c}{d}c_{n+1}c_{n+2} - c_{n+2}^2 - \frac{1}{d}c_{n+1}^2 \\
=& c_{n+2}(-\frac{c}{d}c_{n+1} - c_{n+2}) - \frac{1}{d}c_{n+1}^2 \\
=& \frac{1}{d}c_{n+2}c_n - c_{n+1}^2) = \cdots = \frac{1}{d^{n+1}}(c_2c_0 - c_1^2).
\end{aligned}$$

Finally, if $c_2c_0 - c_1^2 \neq 0 \Longleftrightarrow abc - b^2 - a^2d \neq 0$

$$\det\begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix} \Big/ \det\begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix} = d$$

else $(abc - b^2 - a^2d = 0)$

$$\det\begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix} = \det\begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix} = 0$$

and the ratio is not defined.

**Solution # 2** (by Ashish Rao, Graduate student, ECE; edited by the Panel)

We start with

$$(az+b) = (z^2+cz+d)(c_0+c_1z+c_2z^2+\cdots+c_nz^n+\dots).$$

Comparing the coefficients of $z^{n+2}$ and $z^{n+3}$ for $n \geq 0$ on both sides,

$$\begin{aligned} 0 &= c_n + c\cdot c_{n+1} + d\cdot c_{n+2}, \\ 0 &= c_{n+1} + c\cdot c_{n+2} + d\cdot c_{n+3}. \end{aligned}$$

This is a system of linear equations for $c$ and $d$.
Using Kramer's rule, the solution is:

$$d = \frac{\det\begin{pmatrix} c_n & c_{n+1} \\ c_{n+1} & c_{n+2} \end{pmatrix}}{\det\begin{pmatrix} c_{n+1} & c_{n+2} \\ c_{n+2} & c_{n+3} \end{pmatrix}}.$$

The given ratio is $d$ (it is independent of $n$).
We still need the condition

$$abc - b^2 - a^2d \neq 0$$

to be sure that the denominator is not zero.

Also solved by:

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