PROBLEM OF THE WEEK
Solution of Problem No. 3 (Fall 2009 Series)

Problem: Suppose that \(a_1, a_2, \ldots, a_n\) are real numbers. Obviously if they are all positive, then the \(n\) sums

\[
\sum_{i} a_i, \sum_{i<j} a_i a_j, \sum_{i<j<k} a_i a_j a_k, \ldots, a_1 a_2 \ldots a_n
\]

are all positive. Prove that the converse is also true.

Solution (by Mark Sellke, Klondike Middle School, Indiana)

Let \(\sum a_i = s_1, \sum_{i,j} a_i a_j = s_2\), etc. Note that the \(s_i\)’s correspond to the coefficients of a polynomial of degree \(n\) with roots \(a_1, a_2, \ldots, a_n\): \(p(x) = x^n - x^{n-1}s_1 + x^{n-2}s_2 - \cdots\). As each \(s_i > 0\), the coefficients have alternating signs. Thus, no negative \(r\) can be a root, as \(p(r)\) has terms all of the same sign for \(r < 0\). Also, \(r \neq 0\), as the constant term of the polynomial, \(\pm s_n\), is non–zero. Thus, all real roots are positive, so all \(a_i\)’s are positive, as desired.

The problem was also solved by:

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