PROBLEM OF THE WEEK

Solution of Problem No. 4 (Fall 2009 Series)

Problem: Let $n \geq 5$ be an integer. Show that n is prime if and only if for every decomposition $n = n_1 + n_2 + n_3 + n_4$, where $1 \leq n_1 \leq n_2 \leq n_3 \leq n_4$ and each n_i is an integer, we have $n_1 n_4 \neq n_2 n_3$.

Solution (by Kun-Chieh Wang, Senior, Purdue University)

1. Suppose n is a prime and we could find $n_1, n_2, n_3, n_4 \in \mathbb{N}$ satisfying $n = n_1+n_2+n_3+n_4$, $1 \leq n_1 \leq n_2 \leq n_3 \leq n_4$, and $n_1 n_4 = n_2 n_3$. Let $d_1 = \gcd(n_1, n_2)$, $d_2 = \gcd(n_3, n_4)$, and suppose $n_1 = d_1 p_1$, $n_2 = d_1 p_2$, $n_3 = d_2 q_1$, $n_4 = d_2 q_2$, where $p_1, p_2, q_1, q_2 \in \mathbb{N}$, $\gcd(p_1, p_2) = 1$, $\gcd(q_1, q_2) = 1$.

$$\begin{aligned} n_1 n_4 = n_2 n_3 &\Rightarrow (d_1 p_1)(d_2 q_2) = (d_1 p_2)(d_2 q_1) \\ &\Rightarrow p_1 q_2 = p_2 q_1 \end{aligned}$$

$$\gcd(p_1, p_2) = 1 \text{ and } \gcd(q_1, q_2) = 1 \Rightarrow p_1 | q_1 \text{ and } q_1 | p_1 \Rightarrow p_1 = q_1 \Rightarrow p_2 = q_2$$

$$\begin{aligned} n &= n_1 + n_2 + n_3 + n_4 = d_1 p_1 + d_1 p_2 + d_2 q_1 + d_2 q_2 \\ &= d_1 p_1 + d_1 p_2 + d_2 p_1 + d_2 p_2 \\ &= (d_1 + d_2)(p_1 + p_2) \end{aligned}$$

where $d_1 + d_2 \geq 1 + 1 = 2$, $p_1 + p_2 \geq 1 + 1 = 2 \Rightarrow n$ is a composite number, a contradiction.

2. Suppose n is a composite number. Let $n = ab$ where $a \leq b$, $a, b \in \mathbb{N}$ and $a, b \geq 2$. Then let $n_1 = 1, n_2 = (a-1), n_3 = (b-1), n_4 = (a-1)(b-1)$. Then we have

$$\begin{aligned} &1 \leq n_1 \leq n_2 \leq n_3 \leq n_4, \quad n_1, n_2, n_3, n_4 \in \mathbb{N}, \quad \text{and} \\ &n_1 + n_2 + n_3 + n_4 = (1 + (a-1))(1 + (b-1)) = ab = n. \end{aligned}$$

The problem was also solved by:

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