Chapter 6

Schemes (continued)

6.1 More sheaf theory

A morphism of presheaves of (say) groups $\eta : \mathcal{F} \to \mathcal{G}$ on a space X, is a collection of homomorphisms $\eta_U : \mathcal{F}(U) \to \mathcal{G}(U)$ which commute with restriction: $\eta_V(f|_V) = \eta_U(f)|_V$. If we think of presheaves as functors, as explained earlier, then a morphism is simply a natural transformation. A morphism of sheaves is defined the same way. So we can form a category of presheaves over X, and a subcategory of sheaves on X.

Given a presheaf $\mathcal{F}$ and a point $p \in X$, the stalk

$$\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U)$$

More concretely, an element of $\mathcal{F}_p$ is an equivalence class of a section defined in neighbourhood, where $f \sim f'$ is they agree on a smaller neighbourhood. The equivalence class of f is called the germ of f. There are couple of examples, where the germ can be interpreted in terms of more familiar objects.

Lemma 6.1.1. *Let $\mathcal{O}_{\mathbb{C}}$ be the sheaf of holomorphic functions on $\mathbb{C}$, the stalk $\mathcal{O}_{\mathbb{C},p}$ is isomorphic to the ring of convergent power series at p*

Lemma 6.1.2. $\mathcal{O}_{\operatorname{Spec} R,p} \cong R_p$.

Proof. Let $S = R - p$. Then

$$R_p = S^{-1}R = \varinjlim_{f \notin p} R[f^{-1}] = \varinjlim_{p \in D(f)} \mathcal{O}(D(f))$$

which is the stalk $\mathcal{O}_{\operatorname{Spec} R,p}$. $\square$

We can see that the stalk gives a functor from the category of presheaves to sets, groups etc. Given a presheaf $\mathcal{F}$, define a presheaf

$$\mathcal{F}^+(U) = \{f : U \to \prod_{p \in U} \mid \forall q \in U, \exists q \in V \subseteq U, \exists \phi \in \mathcal{F}(V), \forall p \in U, \phi_p = f(p)\}$$

where restrictions are just restrictions of functions. $\mathcal{F}^+$ is called the sheafification of $\mathcal{F}$. Here is what it does.

Theorem 6.1.3.

1. *$\mathcal{F}^+$ is sheaf,*
2. *there is a morphism $\mathcal{F} \to \mathcal{F}^+$ such that any morphism of $\mathcal{F}$ to a sheaf factors uniquely through $\mathcal{F}$*
3. *The last morphism induces an isomorphism on stalks.*

Given a continuous map $f : X \to Y$ and a presheaf $\mathcal{F}$ on X define the direct image

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$$

with obvious restrictions.

Lemma 6.1.4. *If $\mathcal{F}$ is a sheaf, then so is $f_*\mathcal{F}$.*

6.2 Definition of a scheme

Definition 6.2.1. *A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ all of whose stalks are local rings.*

As a corolloray to lemma 6.1.2, we obtain

Corollary 6.2.2. *An affine scheme is a locally ringed space.*

We now define a morphism of locally ringed spaces. Given locally ringed space $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ a morphism consists of

1. A continuous mapf $F : X \to Y$.
2. A morphism of sheaves of rings

$$F^\# : \mathcal{O}_Y \to \mathcal{O}_X$$

   such that for every $p \in X$, the induced homomorphism

$$\mathcal{O}_{Y,F(p)} \to \mathcal{O}_{X,p}$$

   takes the maximal ideal to the maximal ideal (such a homomorphism is called local).

There is a lot to understand here. The role of the map F should be clear in enough. But also need a way to pullback "functions" from Y to X, and this is where $F^\#$ comes in. This gives us a collection of homomorphisms

$$F^\# : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$$

for every open $U \subseteq Y$. If this really came from pulling back functions, we would see that if a function vanishes at $F(p)$, then its pullback vanishes at p, i.e. that $F^\#(m_{F(p)}) \subseteq m_p$. We impose this as an axiom. Although a morphism is really a pair $(F, F^\#)$, we usually just refer to it as F.

Example 6.2.3. *A regular map of quasiprojective varieties $F : X \to Y$ gives a morphism of locally ringed spaces where we take $F^\# = F^*$.*

Theorem 6.2.4. *Let $f : R \to S$ be homomorphism. Then there is a morphism of locally ringed spaces*

$$F : \operatorname{Spec} S \to \operatorname{Spec} R$$

where $F(p) = f^{-1}p$, and

$$F^\# : \mathcal{O}_{\operatorname{Spec} S}(D(f(r)) \to \mathcal{O}_{\operatorname{Spec} R}(D(r))$$

can be identified with the natural maps

$$R[1/r] \to S[1/f(r)]$$

Conversely, any map of locally ringed spaces from $\operatorname{Spec} S \to \operatorname{Spec} R$ *arises this way from a unique f.*

We refer to Hartshorne for the details. The collection of locally ringed spaces and morphisms form a category.

Corollary 6.2.5 (Duality)**.** *The category of affine schemes is antiequivalent to the category of commutative rings.*

We can define an isomorphism to locally ringed spaces to be a morphism such that F is a homeomorphism and $\mathcal{F}^\#$ is an isomorphism of sheaves.

Definition 6.2.6. *A scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme. More precisely, there exists an open covering $\{U_i\}$, such that $(U_i, \mathcal{O}_X|_{U_i})$ is isomorphic to an affine scheme. Here $\mathcal{O}_X|_{U_i}$ is the sheaf defined by $\mathcal{O}_X|_{U_i}(U) = \mathcal{O}_X(U)$ for $U \subset U_i$.*

We can construct examples as follows.

Lemma 6.2.7. *Let $X = \operatorname{Spec} R$, and $U \subset X$ open. Then $(U, \mathcal{O}_X|_U)$ is a scheme.*

Proof. We can cover U by sets of the form

$$D(f) = \{p \in \operatorname{Spec} R \mid f \notin p\} = \operatorname{Spec} R[1/f]$$

We can see that $D(f) \cong \operatorname{Spec} R[1/f]$ as schemes. Therefore U is locally isomorphic to an affine scheme. □

6.3 Projective space over a ring

Given commutative ring R, we give two constructions of $\mathbb{P}^1_R$. The first is by gluing. Given two topological spaces X_1, X_2, with open sets $U_i \subseteq X_i$, and a homemorphism $\phi : U_1 \cong U_2$, we can form a new space $X = X_1 \cup_\phi X_2$ as follows. First take the disjoint union $X_1 \coprod X_2$. Then form the equivalence relation $\sim$

generated by $x \sim \phi(x)$. Let $X = X_1 \coprod X_2/\sim$ with quotient topology. Open sets $V \subset X$ are obtained by gluing $V_1 \cup_\phi V_2$, where $V_i \subset X_i$ are open/ Now suppose that the X_i are ringed spaces. Define

$$\mathcal{O}_X(V) = \{(r_1, r_2) \in \mathcal{O}_{X_1}(V_1) \times \mathcal{O}_{X_2}(V_2) \mid r_1|_{V_1 \cap U_1} = r_2|_{V_2 \cap U_2}\}$$

Lemma 6.3.1. *If X_i are schemes, then so is X.*

Set $X_1 = \operatorname{Spec} R[x]$ and $X_2 = \operatorname{Spec} R[y]$. These are two copies of the affine line. Define $U_1 = \operatorname{Spec} R[x, x^{-1}]$ and $U_2 = \operatorname{Spec} R[y, y^{-1}]$. Define $\phi : U_1 \cong U_2$ by sending $x \mapsto y^{-1}$. Then glue to get $\mathbb{P}^1_R = X_1 \cup_\phi X_2$.

In principle, we can define $\mathbb{P}^n_R$ by gluing, but it's somewhat messier because we need to glue several schemes. Instead, we use different construction, which is quite useful by itself. Start with a graded commutative ring

$$S = S_0 \oplus S_1 \oplus \dots$$

and set

$$S_+ = S_1 \oplus S_2 \oplus \dots$$

Note that S_0 is actually a subring, and S_+ is an ideal. Let's assume $S_0 = R$. The key example is where $S = R[x_0, \dots, x_n]$ Let $\operatorname{Proj} S$ be the set of homogeneous prime ideals which don't contain S_+. Given a homogeneous element f, i.e. an element of some S_i, let

$$D_+(f) = \{p \in \operatorname{Proj} S \mid f \notin p\}$$

This gives a basis for a topology on $\operatorname{Proj} S$. For f homogeneous, let $S_{(f)} \subset S[1/f]$ generated by ratios g/f^j, where g and f^j are homogenous of the same degree. For example,

$$R[x_0, \dots, x_n]_{(x_i)} = R[x_0/x_i, x_1/x_i, \dots]$$

Theorem 6.3.2. *There exists a sheaf of commutative rings on $X = \operatorname{Proj} S$, such that*

$$\mathcal{O}_X(D_+(f)) = S_{(f)}$$

and restrictions are correspond to natural maps.

Again referring to Hartshorne for details. One consequence of this, $(D_+(f), \mathcal{O}_X|_{D_+(f)})$ is isomorphic to $\operatorname{Spec} S_{(f)}$. Therefore

Corollary 6.3.3. $\operatorname{Proj} S$ *is a scheme.*

We define

$$\mathbb{P}^n_R = \operatorname{Proj} R[x_0, \dots, x_n]$$

Let us compare this to old definition, when $R = k$ is an algebraically closed field. The maximal elements of $\operatorname{Proj} k[x_0, \dots x_n]$, with respect to inclusion, are the ideals of lines in k^{n+1}. These are exactly the closed points. Thus we see that the old version of $\mathbb{P}^n_k$ is the set of closed points of the new version.

6.4 Exercises

Exercise 6.4.1.

1. *Prove lemma 6.1.4.*

2. *Let $i : Y \subset X$ be the inclusion of a closed subset. Given a sheaf $\mathcal{F}$ on Y, calculate the stalks of $i_*\mathcal{F}$ for points in Y and outside Y. You can assume that $\mathcal{F}(\emptyset) = 0$.*

3. *Given a scheme $(X, \mathcal{O}_X)$, and an open set $U \subset X$, check that $(U, \mathcal{O}_X|_U)$ is a scheme. Further that the inclusion $i : U \to X$ can be extended to a morphism of schemes* in a natural way.

4. *A closed immersion of schemes $f : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism f such that,*

   (a) *The underlying map of spaces is injective, and identifies Y with a closed subset of X.*

   (b) *The homomorphism $\mathcal{O}_{X,f(p)} \to \mathcal{O}_{Y,p}$ is surjective.*

   Show that for any commutative ring R and ideal I, the natural map $\operatorname{Spec} R/I \to \operatorname{Spec} R$ is a closed immersion.