Lectures on Grothendieck Duality

II: Derived Hom-Tensor adjointness. Local duality.

Joseph Lipman

Purdue University
Department of Mathematics

lipman@math.purdue.edu

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Outline

1. Derived functors

$Q_{\mathcal{A}} \colon \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ denotes the canonical functor from the homotopy category of an abelian category $\mathcal{A}$ to its derived category.
Let $\mathcal{A}_1$, $\mathcal{A}_2$ be abelian categories, and set $Q_i := Q_{\mathcal{A}_i}$.
Let $\gamma \colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{K}(\mathcal{A}_2)$ be a Δ-functor.

A right-derived functor $(\mathbf{R}\gamma, \zeta)$ of γ consists of a Δ-functor $\mathbf{R}\gamma \colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ and a Δ-functorial map $\zeta \colon Q_2\gamma \to \mathbf{R}\gamma Q_1$ such that every Δ-functorial map $Q_2\gamma \to \Gamma$ where $\Gamma \colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ takes quasi-isomorphisms to isomorphisms, factors uniquely as

$$Q_2\gamma \xrightarrow{\zeta} \mathbf{R}\gamma Q_1 \to \Gamma.$$

In other words, in the category whose objects are functorial maps from $Q_2\gamma$ to variable Γ as above, the map $\zeta(E) \colon Q_2\gamma \to \mathbf{R}\gamma Q_1$ is an *initial object*, and thus it is *unique up to canonical isomorphism.*

Dually:

A left-derived functor $(\mathbf{R}\gamma, \xi)$ of γ consists of a Δ-functor $\mathbf{L}\gamma\colon \mathbf{D}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ and a Δ-functorial map $\xi\colon \mathbf{L}\gamma\, Q_1 \to Q_2\gamma$ such that every Δ-functorial map $\Gamma \to Q_2\gamma$ where $\Gamma\colon \mathbf{K}(\mathcal{A}_1) \to \mathbf{D}(\mathcal{A}_2)$ takes quasi-isomorphisms to isomorphisms, factors uniquely as

$$\Gamma \to \mathbf{L}\gamma\, Q_1 \xrightarrow{\xi} Q_2\gamma.$$

Here ξ is a *final object* in the appropriate category of functorial maps.

Tensor product

We've already seen some right-derived functors, $\mathbf{R}\Gamma_I(-)$ and $\mathbf{R}\mathrm{Hom}(-,-)$. Describe next an important example of a left-derived functor.

The tensor product $C \otimes_R D$ of two R-complexes is such that

$$(C \otimes_R D)^n = \oplus_{i+j=n} C^i \otimes_R D^j,$$

the differential $d^n_\otimes \colon (C \otimes_R D)^n \to (C \otimes_R D)^{n+1}$ being determined by

$$d^n_\otimes(x \otimes y) = d^i_C x \otimes y + (-1)^i x \otimes d^j_D y \qquad (x \in C^i,\ y \in D^j).$$

Fixing D, we get a functor $\gamma_D := - \otimes_R D \colon \mathbf{K}(R) \to \mathbf{K}(R)$, that, together with $\theta :=$ the identity map of $C[1] \otimes_R D = (C \otimes_R D)[1]$, is a Δ-functor.

There is an isomorphism $\rho\colon \gamma'_C(D) := C \otimes_R D \xrightarrow{\ \sim\ } D \otimes_R C = \gamma_C D$ taking $x \otimes y$ to $(-1)^{ij} y \otimes x$.

There is a then a unique Δ-functor (γ'_C, θ') such that ρ is Δ-functorial.

The map $\theta'\colon \gamma'_C(D[1]) \xrightarrow{\ \sim\ } \gamma'_C(D)[1]$ is not the identity: its restriction to $C^i \otimes_R D^j$ is multiplication by $(-1)^i$.

q-flat resolutions

One gets a left-derived functor $-\underline{\otimes}_R D$ of γ_D as follows:

An R-complex F is q-flat if for every exact R-complex E (i.e., $\mathrm{H}^i E = 0$ for all i), $F \otimes_R E$ is exact too.

Equivalently: *the functor $F \otimes_R -$ preserves quasi-isomorphism.*

(By the exactness of the homology sequence of a triangle, a map of complexes is a quasi-isomorphism if and only if its cone is exact, and tensoring with F "commutes" with forming cones.)

For example, any bounded-above (i.e., vanishing above some degree) flat complex is q-flat.

Every R-complex C has a q-flat resolution, i.e., there is a q-flat complex F equipped with a quasi-isomorphism $F \to C$. This can be constructed as a $\varinjlim$ of bounded-above flat resolutions of truncations of C.

For example, a flat resolution of an R-module M

$$\cdots \to F^{-2} \to F^{-1} \to F^0 \to M \to 0$$

can be viewed as a q-flat resolution of M (as a complex).

Left-derived tensor product

After choosing for each C a q-flat resolution $F_C \to C$, one shows there exists a left-derived functor $- \underline{\underline{\otimes}}_R D$ of γ_D with

$$C \underline{\underline{\otimes}}_R D = F_C \otimes_R D$$

If $F_D \to D$ is a q-flat resolution, there are natural $\mathbf{D}(R)$-isomorphisms

$$C \otimes_R F_D \xleftarrow{\ \sim\ } F_C \otimes_R F_D \xrightarrow{\ \sim\ } F_C \otimes_R D,$$

so any of these complexes could be used to define $C \underline{\underline{\otimes}}_R D$.

Using $F_C \otimes_R F_D$ one can, as before, make $C \underline{\underline{\otimes}}_R D$ into a Δ-functor of both variables C and D. As such, it has a initial-object property as above, but with respect to two-variable functors.

Taking homology produces the (hyper)tor functors

$$\mathrm{Tor}_i(C, D) = \mathrm{H}^{-i}(C \underline{\underline{\otimes}}_R D).$$

2. Hom-Tensor adjunction

Relations between Ext and Tor—e.g., as we'll see, Local Duality—are neatly encapsulated by a derived-category upgrade of the basic adjoint associativity relation between Hom and $\otimes$.

For R-modules E, F, G, adjoint associativity is the isomorphism

$$\mathrm{Hom}_R(E \otimes_R F,\, G) \xrightarrow{\sim} \mathrm{Hom}_R\big(E, \mathrm{Hom}_R(F, G)\big)$$

that takes $\phi\colon E \otimes_R F \to G$ to $\phi'\colon E \to \mathrm{Hom}_R(F, G)$ where

$$[\phi'(e)](f) = \phi(e \otimes f) \qquad (e \in E,\ f \in F).$$

More generally, with $\varphi\colon R \to S$ a homomorphism of commutative rings, E, F, S-complexes and G an R-complex, $\exists$ an isomorphism of S-complexes

$$\mathrm{Hom}^\bullet_R(E \otimes_S F,\, G) \xrightarrow{\sim} \mathrm{Hom}^\bullet_S\big(E, \mathrm{Hom}^\bullet_R(F, G)\big) \qquad \text{(adj)}$$

that in degree n takes a family $\big(\phi_{ij}\colon E^i \otimes_S F^j \to G^{i+j+n}\big)$ to the family $\big(\phi'_i\colon E^i \to \mathrm{Hom}^{i+n}_R(F, G)\big)$ with $\phi'_i(e) = \big(\phi'_{ij}(e)\colon F_j \to G^{i+j+n}\big)$ where

$$[\phi'_{ij}(e)](f) = \phi_{ij}(e \otimes f) \qquad (e \in E^i,\ f \in F^j).$$

Derived adjoint associativity

With $\varphi\colon R \to S$ as before, let $\varphi_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$ denote the obvious restriction of scalars functor.

For a fixed S-complex E, the functor $\mathrm{Hom}^\bullet_R(E, G)$ from R-complexes G to S-complexes has a right-derived functor from $\mathbf{D}(R)$ to $\mathbf{D}(S)$ (gotten via q-injective resolution of G), denoted $\mathbf{R}\mathrm{Hom}^\bullet_\varphi(E, G)$.

If we replace G in (adj) by a q-injective resolution, and F by a q-flat one, then the S-complex $\mathrm{Hom}^\bullet_R(F, G)$ is easily seen to become q-injective; and consequently (adj) gives a $\mathbf{D}(S)$-isomorphism

$$\alpha(E, F, G)\colon \mathbf{R}\mathrm{Hom}^\bullet_\varphi(E \underline{\otimes}_S F,\, G) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}^\bullet_S\big(E, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(F, G)\big)$$

The map α is Δ-functorial. Showing this requires some additional grinding.

Derived Hom-Tensor adjunction

α does not depend on the choices of resolutions made above: it is canonically characterized by commutativity, for all E, F and G, of the following otherwise natural $\mathbf{D}(S)$-diagram (where $\mathrm{H}^\bullet$ stands for $\mathrm{Hom}^\bullet$):

$$
\begin{tikzcd}
\mathrm{H}^\bullet_R(E \otimes F, G) \arrow[r] \arrow[d, "\text{(adj)}"', "\simeq"] & \mathbf{R}\mathrm{H}^\bullet_R(\varphi_*(E \otimes F), G) \arrow[r] & \mathbf{R}\mathrm{H}^\bullet_R(\varphi_*(E \underline{\underline{\otimes}} F), G) \arrow[d, "\simeq"', "\alpha"] \\
\mathrm{H}^\bullet_S\big(E, \mathrm{H}^\bullet_R(F, G)\big) \arrow[r] & \mathbf{R}\mathrm{H}^\bullet_S\big(E, \mathrm{H}^\bullet_R(F, G)\big) \arrow[r] & \mathbf{R}\mathrm{H}^\bullet_S\big(E, \mathbf{R}\mathrm{H}^\bullet_R(\varphi_* F, G)\big)
\end{tikzcd}
$$

Application of the functor H^0 to α yields a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}\big(\varphi_*(E \underline{\underline{\otimes}}_S F), G\big) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(S)}\big(E, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(F, G)\big),$$

Thus:

For fixed $F \in \mathbf{D}(S)$, there is a natural adjunction between the functors

$\varphi_*(- \underline{\underline{\otimes}}_S F)\colon \mathbf{D}(S) \to \mathbf{D}(R)$ and $\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* F, -)\colon \mathbf{D}(R) \to \mathbf{D}(S)$.

3. Abstract local duality

Recall briefly the connection between $\mathbf{R}\Gamma_I$ and Koszul complexes.

R is a commutative *noetherian* ring; $\otimes := \otimes_R$.

$\mathbf{t} = (t_1, \dots, t_m)$ is a sequence in R, generating the ideal $I := \mathbf{t}R$.

For $t \in R$, let $\mathcal{K}(t)$ be the complex that in degrees 0 and 1 is the usual map from R to the localization R_t, and that vanishes elsewhere.

For any R-complex C, define the "stable" Koszul complex

$$\mathcal{K}(\mathbf{t}) := \mathcal{K}(t_1) \otimes \cdots \otimes \mathcal{K}(t_m), \qquad \mathcal{K}(\mathbf{t}, C) := \mathcal{K}(\mathbf{t}) \otimes C.$$

Since the complex $\mathcal{K}(\mathbf{t})$ is flat and bounded, hence q-flat, $\mathcal{K}(\mathbf{t}, -)$ takes quasi-isomorphisms to quasi-isomorphisms and so may—and will—be regarded as a functor from $\mathbf{D}(R)$ to $\mathbf{D}(R)$.

Given a q-injective resolution $C \to E_C$ we have for $E = E_C^j$ $(j \in \mathbb{Z})$,

$$\Gamma_I E = \ker\big(\mathcal{K}^0(\mathbf{t}, E) = E \to \oplus_{i=1}^m E_{t_i} = \mathcal{K}^1(\mathbf{t}, E)\big),$$

whence a $\mathbf{D}(R)$-map

$$\delta(C)\colon \mathbf{R}\Gamma_I C = \Gamma_I E_C \hookrightarrow \mathcal{K}(\mathbf{t}, E_C) \cong \mathcal{K}(\mathbf{t}, C).$$

The following proposition is a key to many properties of Γ_I.
(Details in §3 of "Lectures on Local Cohomology...")

Proposition

The $\mathbf{D}(R)$-map $\delta(C)$ is a functorial *isomorphism*

$$\mathbf{R}\Gamma_I C \xrightarrow{\sim} \mathcal{K}(\mathbf{t}, C).$$

Since $\mathcal{K}(\mathbf{t}, C) = \mathcal{K}(\mathbf{t}, R) \otimes C$ and $\mathcal{K}(\mathbf{t}, R) \cong \mathbf{R}\Gamma_I R$ is q-flat, therefore:

Corollary

There is a functorial $\mathbf{D}(R)$ isomorphism

$$\mathbf{R}\Gamma_I C \xrightarrow{\sim} (\mathbf{R}\Gamma_I R) \underline{\underline{\otimes}} C.$$

Taking homology, one gets:

$$\mathrm{H}^i_I(C) = \mathrm{H}^i\mathbf{R}\Gamma_I C \cong \mathrm{Tor}_{-i}(\mathbf{R}\Gamma_I R, C) \qquad (i \in \mathbb{Z}).$$

Local duality

Let J be an S-ideal. Let $\varphi_J^\#\colon \mathbf{D}(R) \to \mathbf{D}(S)$ be the functor

$$\begin{aligned}\varphi_J^\#(G) &:= \mathbf{R}\mathrm{Hom}^\bullet_\varphi(\mathbf{R}\Gamma_J S, G)\\ &\cong \mathbf{R}\mathrm{Hom}^\bullet_S\big(\mathbf{R}\Gamma_J S, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G)\big) \qquad \big(G \in \mathbf{D}(R)\big),\end{aligned}$$

the isomorphism being derived by setting $E = \mathbf{R}\Gamma_J S$ and $F = S$ in the derived adjoint associativity isomorphism

$$\alpha(E, F, G)\colon \mathbf{R}\mathrm{Hom}^\bullet_\varphi(E \underline{\underline{\otimes}}_S F,\, G) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_S\big(E, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(F, G)\big).$$

For $E \in \mathbf{D}(S)$ and $G \in \mathbf{D}(R)$, one has then functorial $\mathbf{D}(S)$-isomorphisms

$$\mathbf{R}\mathrm{Hom}^\bullet_\varphi(\mathbf{R}\Gamma_J E,\, G) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_\varphi(E \underline{\underline{\otimes}}_S \mathbf{R}\Gamma_J S,\, G) \xrightarrow[\alpha]{\ \sim\ } \mathbf{R}\mathrm{Hom}^\bullet_S\big(E, \varphi_J^\# G\big).$$

Application of the functor $\mathrm{H}^0\varphi_*$ produces the local duality isomorphism

$$\mathrm{Hom}_{\mathbf{D}(R)}(\varphi_*\mathbf{R}\Gamma_J E,\, G) \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathbf{D}(S)}(E, \varphi_J^\# G),$$

an adjunction between the functors $\varphi_*\mathbf{R}\Gamma_J$ and $\varphi_J^\#$.

4. Concrete local duality

Henceforth, all rings are noetherian as well as commutative.

Concrete versions of local duality convey more information about $\varphi_J^{\#}$.

Suppose, for example, that S is *module-finite* over R, and let $G \in \mathbf{D}_{\mathrm{c}}(R)$, i.e., each homology module of $G \in \mathbf{D}(R)$ is finitely generated. Suppose also that $\mathrm{Ext}^i_R(S, G)$ is a finitely-generated R-module for all $i \in \mathbb{Z}$, i.e., $\mathbf{R}\mathrm{Hom}^\bullet_R(\varphi_* S, G) \in \mathbf{D}_{\mathrm{c}}(R)$. (This holds, e.g., if $\mathrm{H}^i G = 0$ for all $i \ll 0$.)

Then $\mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G) \in \mathbf{D}_{\mathrm{c}}(S)$, since, as is easily seen,

$$\varphi_* \mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G) \cong \mathbf{R}\mathrm{Hom}_R(\varphi_* S, G) \in \mathbf{D}_{\mathrm{c}}(R).$$

Now Greenlees-May duality (= Grothendieck duality for the natural map $\mathrm{Spec}(\hat{S}) \to \mathrm{Spec}(S)$, with $\hat{S}$ the J-adic completion of S), gives

$$\mathbf{R}\mathrm{Hom}^\bullet_S(\mathbf{R}\Gamma_J S, F) \cong F \otimes_S \hat{S} \qquad (F \in \mathbf{D}_{\mathrm{c}}(S)).$$

In particular:

$$\varphi_J^{\#} G = \mathbf{R}\mathrm{Hom}^\bullet_S(\mathbf{R}\Gamma_J S, \mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G)) \cong \mathbf{R}\mathrm{Hom}^\bullet_\varphi(S, G) \otimes_S \hat{S}.$$

More in particular, for $S = R$ and $\varphi = \mathrm{id}$ (the identity map) one gets

$$\mathrm{id}_J^{\#}G = G \otimes_R \hat{R} \qquad \big(G \in \mathbf{D}_{\mathrm{c}}(R)\big).$$

Specialize further to where R is local, $\varphi = \mathrm{id}$, $J = \mathfrak{m}$, the maximal ideal of R, and $G \in \mathbf{D}_{\mathrm{c}}(R)$ is a normalized dualizing complex (exists if R is a homomorphic image of a Gorenstein local ring), so that in $\mathbf{D}(R)$, $\mathcal{I} := \mathbf{R}\Gamma_{\mathfrak{m}} G$ is an R-injective hull of the residue field $R/\mathfrak{m}$.

Then there is a natural isomorphism

$$\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_{\mathfrak{m}}E, \mathcal{I}) = \mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_{\mathfrak{m}}E, \mathbf{R}\Gamma_{\mathfrak{m}}G) \cong \mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_{\mathfrak{m}}E, G)$$

Substitution into the local duality isomorphism gives, for all $E \in \mathbf{D}(R)$,

$$\mathbf{R}\mathrm{Hom}^{\bullet}_R(\mathbf{R}\Gamma_{\mathfrak{m}}E, \mathcal{I}) \xrightarrow{\ \sim\ } \mathbf{R}\mathrm{Hom}^{\bullet}_R(E, \mathrm{id}_J^{\#}G) = \mathbf{R}\mathrm{Hom}^{\bullet}_R(E, G \otimes_R \hat{R}).$$

For $E \in \mathbf{D}_{\mathrm{c}}(R)$ this is just classical local duality, modulo Matlis duality.

More familiar local duality

Applying homology H^{-i} one gets the duality isomorphism

$$\mathrm{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}E,\mathcal{I}) \xrightarrow{\ \sim\ } \mathrm{Ext}^{-i}_R(E, G\otimes_R \hat{R}).$$

Suppose R Cohen-Macaulay, i.e., there's an $\mathfrak{m}$-primary ideal generated by an R-regular sequence of length $d := \dim(R)$. Then $\mathrm{H}^i_{\mathfrak{m}}R = 0$ for $i \neq d$. Since $\hat{R}$ is R-flat, the preceding isomorphism now yields, for $i \neq d$,

$$\begin{aligned}0 = \mathrm{Ext}^{-i}_R(R, G\otimes_R \hat{R}) = \mathrm{H}^{-i}\mathbf{R}\mathrm{Hom}^{\bullet}(R, G\otimes_R \hat{R}) = \mathrm{H}^{-i}(G\otimes_R \hat{R})\\ = (\mathrm{H}^{-i}G)\otimes_R \hat{R}.\end{aligned}$$

Hence $H^{-i}G = 0$, so there is a derived-category isomorphism $G \cong \omega[d]$ where $\omega := \mathrm{H}^{-d}G$, a canonical module of R.

Thus, when R is Cohen-Macaulay local duality takes the familiar form

$$\mathrm{Hom}_R(\mathrm{H}^i_{\mathfrak{m}}E,\mathcal{I}) \xrightarrow{\ \sim\ } \mathrm{Ext}^{d-i}_R(E, \hat{\omega}).$$

5. Residues and duality for power series rings

Another situation in which $\varphi_J^\#$ can be described concretely is when φ is the inclusion of R into a power-series ring $S := R[[\mathbf{t}]] := R[[t_1, \dots, t_m]]$, and J is the ideal $\mathbf{t}S = (t_1, \dots, t_m)S$.

There exist an S-module $\hat\Omega_{S/R}$ and an R-derivation $d\colon S \to \hat\Omega_{S/R}$ such that $(dt_1, \dots, dt_m)$ is a free S-basis of $\hat\Omega_{S/R}$, characterized by the universal property that for any finitely-generated S-module M and R-derivation $D\colon S \to M$ there is a unique S-linear map $\delta\colon \hat\Omega_{S/R} \to M$ such that $D = \delta d$.

Let $\hat\Omega^m$ $(m > 0)$ be the m-th exterior power of $\hat\Omega_{S/R}$, a free rank-one S-module with basis $dt_1 \wedge dt_2 \cdots \wedge dt_m$.

Then (fact) there is a canonical functorial isomorphism

$$\varphi_J^\# G \xrightarrow{\ \sim\ } G \otimes \hat\Omega^m[m] \qquad \big(G \in \mathbf{D}_{\mathrm{c}}(R)\big).$$

Residue map

There is a natural surjection

$$\pi\colon (\hat{\Omega}^m)_{t_1 t_2 \cdots t_m} = \mathcal{K}^m(\mathbf{t}, \hat{\Omega}^m) \twoheadrightarrow \mathrm{H}^m\mathcal{K}(\mathbf{t}, \hat{\Omega}^m) = \mathrm{H}^m_J \hat{\Omega}^m$$

For $\nu \in \hat{\Omega}^m$ and nonnegative integers $n_1, \ldots, n_m$, set

$$\begin{bmatrix} \nu \\ t_1^{n_1}, \ldots, t_m^{n_m} \end{bmatrix} := \pi\left(\frac{\nu}{t_1^{n_1}\cdots t_m^{n_m}}\right).$$

Theorem

There is a canonical (i.e., depending only on the topological R-algebra S) residue map

$$\mathrm{res}_{S/R}\colon \mathrm{H}^m_J \hat{\Omega}^m \to R,$$

such that

$$\mathrm{res}_{S/R} \begin{bmatrix} dt_1 \cdots dt_m \\ t_1^{n_1}, \ldots, t_m^{n_m} \end{bmatrix} = \begin{cases} 1 & \text{if } n_1 = \cdots = n_m = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Canonical local duality

As a concrete realization of the abstract local duality theorem, one has, in the preceding situation, an affine version of Serre duality:

Theorem

There is, for S-modules E, a canonical functorial isomorphism

$$\mathrm{Hom}_R(\mathrm{H}^m_J E, R) \xrightarrow{\ \sim\ } \mathrm{Hom}_S(E, \hat{\Omega}^m)$$

that for $E = \hat{\Omega}^m$ takes $\mathrm{res}_{S/R}$ to the identity map of $\hat{\Omega}^m$.

In other words:

The functor $\mathrm{Hom}_R(\mathrm{H}^m_J E, R)$ of S-modules E is represented by $(\hat{\Omega}^m, \mathrm{res}_{S/R})$.

Proofs of the foregoing statements are in "Lectures. . . ," §5.

Wrap-up

It has been illustrated that duality theory is a (gold) coin with two faces, the abstract and the concrete.

Typically, concrete theorems are more striking, and harder to prove directly than their abstract counterparts; but passing from abstract to concrete is not easy. Indeed, it is one of the most challenging aspects of the area.